XX ESCUELA VENEZOLANA DE MATEMÁTICAS

SEMI-FREDHOLM OPERATORS, PERTURBATION THEORY AND LOCALIZED SVEP

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To my children Marco, Caterina and Adriana

Contents

Preface

These notes are intended to provide an introduction to the spectral theory of bounded linear operators defined on Banach spaces. Central items of interest include Fredholm operators and semi-Fredholm operators, but we also consider some of their generalizations which have been studied in recent years.

The spectral theory of operators is an important part of functional analysis which has application in several areas of modern mathematical analysis and physics, for instance in differential and integral equation, as well as quantum theory.

In a sense operators may be studied from two different points of view. One aspect stresses the membership in a certain structured class of operators, such as a particular operator ideal in the sense of Pietsch [**92**]. The other point of view concentrates on the investigation of certain distinguished parts of the spectrum for an individual operator.

The classical study of the spectral structure of an operator has recently been enriched by the development of some powerful new methods for the local analysis of the spectrum. A basic concept in local spectral theory is given by the single-valued extension property (SVEP). This property was introduced by Dunford as a tool for the general theory of spectral operators, as documented by the extensive treatment in the monograph by Dunford and Schwarz [**50**]. SVEP also plays an important role in the recent books of Laursen and Neumann [**76**] and of Aiena [**1**]. One of the purposes of these notes is that of relating a localized version of this property to the study of semi-Fredholm operators and some of their generalizations.

Our book consists of four chapters, whose architecture we shortly describe in the sequel. In Chapter 1, we preliminarily develop the classical theory of semi-Fredholm operators, and establish, in particular, the most important results of the so-called Riesz–Schauder theory of compact operators. A section of this chapter is also devoted to the study of perturbation properties of semi-Fredholm operators.

Chapter 2 deals with the elegant interaction between the localized SVEP and Fredholm theory. This interaction is studied in the more general context of operators of Kato type. In this chapter we also introduce Riesz, Weyl and Browder operators.

The third chapter addresses the study of some perturbation ideals which occur in Fredholm theory. In particular we study the ideal of inessential operators, the ideal of strictly singular operators and the ideal of strictly cosingular operators.

The fourth chapter deals with spectral theory, we focus on the study of several spectra that originating from Fredholm theory. We shall also introduce some special classes of operators having nice spectral properties. These operators include those for which Browder's theorem and Weyl's theorem hold. We also consider some variations of both theorems and the corresponding perturbation theory.

All chapters are concluded by a section where we give further information and discuss some of the more recent developments in the theory previously developed. In general, all the results established in these final sections are presented without proofs. However, we always give appropriate references to the original sources, where the reader can find the relative details.

Of course, it is not possible to make a presentation such as this one entirely self-contained. We require some modest prerequisites from functional analysis and operator theory that the reader can find in the classical texts of functional analysis, as, for instance the book of Heuser [**68**] or the book of Lay and Taylor [**75**].

We therefore have the hope, although it may only be a unreasonable wish, that these notes are accessible to newcomers and graduate students of mathematics with a standard background in analysis.

A considerable part of the content of these notes corresponds to research activities developed during several visits at the departments of mathematics of some universities of Venezuela, as ULA (Merida), UDO (Cumaná) and UCLA (Barquisimeto). In particular, most of the content of Chapter 2 and of Chapter 4 corresponds to results obtained in collaboration with some colleagues working at these universities, as well as with some of my graduate students. For this reason it has been a great pleasure for me to organize the material of these notes.

These notes also form an extended version of a series of lectures in the Department of Mathematics of the Universidad de Los Andes, Merida (Venezuela), in September 2007. These lectures was given in the framework of the activities of the Escuela Venezolana de Matemáticas. I would like to thank the organizers for inviting to me, and in particular all the members of the various institutions which supported such an event.

Finally, I want to thank my graduate students, Jesús Guillén and Pedro Peña, for reading the material of this book and for very helpful remarks.

CHAPTER 1

Semi-Fredholm operators

In this chapter we introduce the class of semi-Fredholm operators $\Phi_\pm(X,Y)$, acting between Banach spaces, and some other classes of operators related to them. We concern with the algebraic and topological structure of $\Phi_\pm(X,Y)$, as well as with some perturbation properties. Before we give some basic informations on the operators which have closed range and, successively, we establish the basic relationships between some of the classical quantities of operator theory, as the ascent, the descent, the nullity and the deficiency of an operator. In the third section we develop the so called Riesz–Schauder theory for compact linear operators acting on Banach spaces and, successively, we establish the main perturbation results for semi-Fredholm and Fredholm operators. In the last section we give information on some other classes of operators, introduced and studied more recently, which satisfy some of the properties already observed for semi-Fredholm operators.

1. Operators with closed range

If X, Y are Banach spaces, by $L(X,Y)$ we denote the Banach space of all bounded linear operators from X into Y. Although many of the results in these notes are valid for real Banach spaces, we always assume that all Banach spaces are *complex* infinite-dimensional Banach spaces. Recall that if $T \in L(X,Y)$, the norm of T is defined by

$$\|T\| := \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

If $X = Y$ we write $L(X) := L(X,X)$. By $X^* := L(X,\mathbb{C})$ we denote the *dual* of X. If $T \in L(X,Y)$ by $T^* \in L(Y^*,X^*)$ we denote the *dual operator* of T defined by

$$(T^*f)(x) := f(Tx) \quad \text{for all } x \in X,\ f \in Y^*.$$

The identity operator on X will be denoted by I_X, or simply I if no confusion can arise. Given a bounded operator $T \in L(X,Y)$, the *kernel* of T is the set

$$\ker T := \{x \in X : Tx = 0\},$$

while the *range* of T is denoted by $T(X)$. A classical result of functional analysis states that for every $T \in L(X)$ then

$$T(X) \quad \text{is closed} \Leftrightarrow T^*(X^*) \quad \text{is closed},$$

see Theorem 97.1 of [**68**]. The property of $T(X)$ being closed may be characterized by means of a suitable number associated with T.

Definition 1.1. *If $T \in L(X,Y)$, X, Y Banach spaces, the* reduced minimal modulus *of T is defined to be*

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}.$$

Formally we set $\gamma(0) = \infty$. It easily seen that if T is bijective then $\gamma(T) = \frac{1}{\|T^{-1}\|}$. In fact, if T is bijective then $\operatorname{dist}(x, \ker T) = \operatorname{dist}(x, \{0\}) = \|x\|$, thus if $Tx = y$,

$$\begin{aligned}
\gamma(T) &= \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|} = (\sup_{x \neq 0} \frac{\|x\|}{\|Tx\|})^{-1} \\
&= (\sup_{y \neq 0} \frac{\|T^{-1}y\|}{\|y\|})^{-1} = \frac{1}{\|T^{-1}\|}.
\end{aligned}$$

Theorem 1.2. *Let $T \in L(X,Y)$, X and Y Banach spaces. Then we have*

(i) $\gamma(T) > 0$ *if and only if $T(X)$ is closed.*

(ii) $\gamma(T) = \gamma(T^*)$.

Proof (i) The statement is clear if $T = 0$. Suppose that $T \neq 0$. Let $\overline{X} := X/\ker T$ and denote by $\overline{T} : \overline{X} \to Y$ the continuous injection corresponding to T, defined by

$$\overline{T}\,\overline{x} := Tx \quad \text{for every } x \in \overline{x}.$$

It is easy to see that $\overline{T}(\overline{X}) = T(X)$. But it is known that $\overline{T}(\overline{X})$ is closed if and only if $\overline{T}$ admits a continuous inverse, i.e., there exists a constant $\delta > 0$ such that $\|\overline{T}\overline{x}\| \geq \delta\|\overline{x}\|$, for every $x \in X$. From the equality

$$\gamma(T) = \inf_{\overline{x} \neq \overline{0}} \frac{\|\overline{T}\,\overline{x}\|}{\|\overline{x}\|}$$

we then conclude that $\overline{T}(\overline{X}) = T(X)$ is closed if and only if $\gamma(T) > 0$.

(ii) The assertion is obvious if $\gamma(T) = 0$. Suppose that $\gamma(T) > 0$. Then $T(X)$ is closed. If $\overline{T}_0 : \overline{X} \to T(X)$ is defined by $\overline{T}_0\overline{x} := Tx$ for every $x \in \overline{x}$, then $\gamma(T) = \gamma(\overline{T}_0)$ and $T = J\overline{T}_0 Q$, where $J : T(X) \to Y$ is the natural embedding, $Q : X \to \overline{X}$ is the canonical projection defined by $Qx = \overline{x}$. Clearly, that $\overline{T}_0$ is bijective and from $T = J\overline{T}_0 Q$ it then follows that $T^* = Q^*(\overline{T}_0)^* J^*$. From this it easily follows that

$$\gamma(T) = \frac{1}{\|(\overline{T}_0)^{-1}\|} = \frac{1}{\|(\overline{T}_0{}^*)^{-1}\|} = \gamma(T^*).$$

■

Let M be a subset of a Banach space X. The *annihilator* of M is the closed subspace of X^* defined by

$$M^\perp := \{f \in X^* : f(x) = 0 \text{ for every } x \in M\},$$

while the *pre-annihilator* of a subset W of X^* is the closed subspace of X defined by

$$^\perp W := \{x \in X : f(x) = 0 \text{ for every } f \in W\}.$$

Clearly $^\perp(M^\perp) = M$ if M is closed. Moreover, if M and N are closed linear subspaces of X then $(M + N)^\perp = M^\perp \cap N^\perp$. The dual relation $M^\perp + N^\perp = (M \cap N)^\perp$ is not always true, since $(M \cap N)^\perp$ is always closed but $M^\perp + N^\perp$ need not be closed. However, a classical theorem establishes that

$$M^\perp + N^\perp \text{ is closed in } X^* \Leftrightarrow M + N \text{ is closed in } X,$$

see Kato [**71**, Theorem 4.8, Chapter IV].

The following duality relationships between the kernels and ranges of a bounded operator T on a Banach space and its dual T^* are well known, (the reader can find the proofs for instance in Heuser [**68**, p.135]:

(1) $$\ker T = {}^\perp \overline{T^*(X^*)} \quad \text{and} \quad {}^\perp\ker T^* = \overline{T(X)},$$

and

(2) $$\overline{T(X)}^\perp = \ker T^* \quad \text{and} \quad \overline{T^*(X^*)} \subseteq \ker T^\perp.$$

Note that the last inclusion is, in general, strict. However, a classical result states that the equality holds precisely when T has closed range, see Kato [**71**, Theorem 5.13, Chapter IV].

In the next theorem we establish some basic isomorphisms needed in the sequel.

Theorem 1.3. *Let M be a closed subspace of a Banach space X. Then M^* is isometrically isomorphic to the quotient $X^*/M^\perp$, while $(X/M)^*$ is isometrically isomorphic to $M^\perp$.*

Proof Let $J_M : M \to X$ the natural embedding of M into X. Then the dual ${J_M}^* : X^* \to M^*$ is the operator ${J_M}^*(f) = f|M$, where $f|M$ is the restriction of $f \in X^*$ to M. Clearly, $\ker {J_M}^* = M^\perp$. Define $W : X^*/M^\perp \to M^*$ by

$$W(f + M^\perp) := {J_M}^* f \quad \text{for all } f \in X^*.$$

It is not difficult to show that W is an isometry, so the first assertion is proved.

To show the second statement, denote by $Q_M : X \to X/M$ the canonical quotient map. Then its dual ${Q_M}^* : (X/M)^* \to X^*$ is an isometry and

$${Q_M}^*((X/M)^*) = (\ker Q_M)^\perp = M^\perp,$$

so $(X/M)^*$ is isometrically isomorphic to $M^\perp$. ■

A very important class of operators is the class of injective operators having closed range.

Definition 1.4. *An operator $T \in L(X)$ is said to be* bounded below *if T is injective and has closed range.*

Theorem 1.5. *$T \in L(X, Y)$ is bounded below if and only if there exists $K > 0$ such that*

$$\|Tx\| \geq K\|x\| \quad \textit{for all } x \in X. \tag{3}$$

Proof Indeed, if $\|Tx\| \geq K\|x\|$ for some $K > 0$ and all $x \in X$ then T is injective. Moreover, if (x_n) is a sequence in X for which (Tx_n) converges to $y \in X$ then (x_n) is a Cauchy sequence and hence convergent to some $x \in X$. Since T is continuous then $Tx = y$ and therefore $T(X)$ is closed.

Conversely, if T is injective and $T(X)$ is closed then, from the open mapping theorem, it easily follows that there exists a $K > 0$ for which the inequality (3) holds. ■

The quantity

$$j(T) := \inf_{\|x\|=1} \|Tx\| = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

is called the *injectivity modulus* of T and obviously from (3) we have

$$T \text{ is bounded below} \Leftrightarrow j(T) > 0, \tag{4}$$

and in this case $j(T) = \gamma(T)$. The next result shows that the properties to be bounded below or to be surjective are dual each other.

Theorem 1.6. *Let $T \in L(X)$, X a Banach space. Then:*

(i) *T is surjective (respectively, bounded below) if and only if T^* is bounded below (respectively, surjective);*

(ii) *If T is bounded below (respectively, surjective) then $\lambda I - T$ is bounded below (respectively, surjective) for all $|\lambda| < \gamma(T)$.*

Proof (i) Suppose that T is surjective. Trivially T has closed range and therefore also T^* has closed range. From the equality $\ker T^* = T(X)^\perp = X^\perp = \{0\}$ we conclude that T^* is injective.

Conversely, suppose that T^* is bounded below. Then T^* has closed range and hence by Theorem 1.2 the operator T has also closed range. From the equality $T(X) =^\perp \ker T^* =^\perp \{0\} = X$ we then conclude that T is surjective.

The proof of T being bounded below if and only if T^* is surjective is analogous.

(ii) Suppose that T is injective with closed range. Then $\gamma(T) > 0$ and from definition of $\gamma(T)$ we obtain

$$\gamma(T) \cdot \operatorname{dist}(x, \ker T) = \gamma(T)\|x\| \leq \|Tx\| \quad \text{for all } x \in X.$$

From that we obtain

$$\|(\lambda I - T)x\| \geq \|Tx\| - |\lambda|\|x\| \geq (\gamma(T) - |\lambda|)\|x\|,$$

thus for all $|\lambda| < \gamma(T)$, the operator $\lambda I - T$ is bounded below.

The case that T is surjective follows now easily by considering the adjoint T^*. ■

Theorem 1.7. *Let $T \in L(X)$, X a Banach space, and suppose that there exists a closed subspace Y of X such that $T(X) \oplus Y$ is closed and $T(X) \cap Y = \{0\}$. Then the subspace $T(X)$ is also closed.*

Proof Consider the product space $X \times Y$ under the norm $\|(x, y)\| := \|x\| + \|y\|$, $x \in X$, $y \in Y$. Then $X \times Y$ is a Banach space and the continuous map $S : X \times Y \to X$ defined by $S(x, y) := Tx + y$ has range $S(X \times Y) = T(X) \oplus Y$ closed by assumption. Hence

$$\gamma(S) := \inf_{(x,y) \notin \ker S} \frac{\|S(x, y)\|}{\operatorname{dist}((x, y), \ker S)} > 0.$$

Moreover, $\ker S = \ker T \times \{0\}$, so

$$\mathrm{dist}((x,0), \ker S) = \mathrm{dist}(x, \ker T),$$

and hence

$$\begin{aligned} \|Tx\| &= \|S(x,0)\| \geq \gamma(S)\ \mathrm{dist}((x,0), \ker S)) \\ &= \gamma(S)\ \mathrm{dist}(x, \ker T). \end{aligned}$$

¿From this it follows that $\gamma(T) \geq \gamma(S) > 0$, and this implies that T has closed range. ■

It is obvious that the sum $M + N$ of two linear subspaces M and N of a vector X space is again a linear subspace. If $M \cap N = \{0\}$ then this sum is called the *direct sum* of M and N and will be denoted by $M \oplus N$. In this case for every $z = x + y$ in $M + N$ the components x, y are uniquely determined. If $X = M \oplus N$ then N is called an *algebraic complement* of M. In this case the (Hamel) basis of X is the union of the basis of M with the basis of N. It is obvious that every subspace of a vector space admits at least one algebraic complement. The *codimension* of a subspace M of X is the dimension of every algebraic complement N of M, or equivalently the dimension of the quotient X/M. Note that $\mathrm{codim}\, M = \dim M^\perp$. Indeed, by Theorem 1.3 we have:

$$\mathrm{codim}\, M = \dim X/M = \dim\,(X/M)^* = \dim M^\perp,$$

Theorem 1.7 then yields directly the following important result:

Corollary 1.8. *Let $T \in L(X)$, X a Banach space, and Y a finite-dimensional subspace of X such that $T(X) + Y$ is closed. Then $T(X)$ is closed. In particular, if $T(X)$ has finite codimension then $T(X)$ is closed.*

Proof Let Y_1 be any subspace of Y for which $Y_1 \cap T(X) = \{0\}$ and $T(X) + Y_1 = T(X) + Y$. From the assumption we infer that $T(X) \oplus Y_1$ is closed, so $T(X)$ is closed by Theorem 1.7. The second statements is clear, since every finite-dimensional subspace of a Banach space X is always closed. ■

A particularly important class of endomorphisms are the so-called *projections*. If $X = M \oplus N$ and $x = y + z$, with $x \in M$ and $y \in N$, define $P : X \to M$ by $Px := y$. The linear map P projects X onto M along N. Clearly, $I - P$ projects X onto N along M and we have

$$P(X) = \ker\,(I - P) = M, \quad \ker P = (I - P)(X) = N, \quad \text{with } P^2 = P,$$

i.e. P is an idempotent operator. Suppose now that X is a Banach space. If $X = M \oplus N$ and the projection P is continuous then M is said to be *complemented* and N is said to be a *topological complement* of M. Note that each complemented subspace is closed, but the converse is not true, for instance c_0, the Banach space of all sequences which converge to 0, is a not complemented closed subspace of ℓ_∞, where ℓ_∞ denotes the Banach space of all bounded sequences, see [**86**].

Definition 1.9. *$T \in L(X,Y)$, X a Banach space, is said to be* relatively regular *if there exists an operator $S \in L(Y,X)$ for which*

$$T = TST \quad \textit{and} \quad STS = S.$$

There is no loss of generality if we require in the definition above only $T = TST$. In fact, if $T = TST$ holds then the operator $S' := STS$ will satisfy both the equalities

$$T = TS'T \quad \text{and} \quad S' = S'TS'.$$

We now establish a basic result.

Theorem 1.10. *A bounded operator $T \in L(X,Y)$ is relatively regular if and only if $\ker T$ and $T(X)$ are complemented.*

Proof If $T = TST$ and $STS = S$ then $P := TS \in L(Y)$ and $Q := ST \in L(X)$ are idempotents, hence projections. Indeed

$$(TS)^2 = TSTS = TS \quad \text{and} \quad (ST)^2 = STST = ST.$$

Moreover, from the inclusions

$$T(X) = (TST)(X) \subseteq (TS)(Y) \subseteq T(X),$$

and

$$\ker T \subseteq \ker(ST) \subseteq \ker(STS) = \ker T,$$

we obtain $P(Y) = T(X)$ and $\ker Q = (I_X - Q)(X) = \ker T$.

Conversely, suppose that $\ker T$ and $T(X)$ are complemented in X and Y, respectively. Write $X = \ker T \oplus U$ and $Y = T(X) \oplus V$ and let us denote by P the projection of X onto $\ker T$ along U and by Q_0 the projection of Y onto $T(X)$ along V. Define $T_0 : U \to T(X)$ by $T_0 x = Tx$ for all $x \in U$. Clearly T_0 is bijective. Put $S := {T_0}^{-1}Q_0$. If we represent an arbitrary $x \in X$ in the form $x = y + z$, with $y \in \ker T$ and $z \in U$, we obtain

$$\begin{aligned} STx &= {T_0}^{-1}Q_0T(y+z) = {T_0}^{-1}Q_0Tz \\ &= {T_0}^{-1}Tz = z = x - y = x - Px. \end{aligned}$$

Similarly one obtains $TS = Q_0$. If $Q := I_Y - Q_0$ then

$$ST = I_X - P \quad \text{and} \quad TS = I_Y - Q. \tag{5}$$

If we multiply the first equation in (5) from the left by T we obtain $TST = T$, and analogously multiplying the second equation in (5) from the left by S we obtain $STS = S$. ■

The left, or right, invertible operators may be characterized as follows:

Theorem 1.11. *Let $T \in L(X,Y)$, X and Y Banach spaces.*

(i) *T is injective and $T(X)$ is complemented if and only if there exists $S \in L(Y,X)$ such that $ST = I_X$.*

(ii) *T is surjective and* ker T *is complemented if and only if there exists $S \in L(Y,X)$ such that $TS = I_Y$.*

Proof (i)If $S \in L(Y,X)$ and $ST = I_X$ then $TST = T$, thus T is relatively regular and hence has complemented range, by Theorem 1.10. Clearly, T is injective. Conversely, if T is bounded below and P is a projection of X onto $T(X)$, let $S_0 : T(X) \to X$ be the inverse of T. If $S := S_0 P$ then $ST = I_X$.

(ii) If $TS = I_Y$ then $TST = T$, so T has complemented kernel by Theorem 1.10 and, as it is easy to see, T is onto. Conversely, if T is onto and $X = \ker T \oplus N$ then $T|N : N \to Y$ is bijective. Let $J_N : N \to X$ be the natural embedding and set $S := J_N(T|N)^{-1}$. Clearly, $TS = I_Y$. ■

2. Ascent and descent

The kernels and the ranges of the iterates T^n, $n \in \mathbb{N}$, of a linear operator T defined on a vector space X, form two increasing and decreasing chains, respectively, i.e. the chain of kernels

$$\ker T^0 = \{0\} \subseteq \ker T \subseteq \ker T^2 \subseteq \cdots$$

and the chain of ranges

$$T^0(X) = X \supseteq T(X) \supseteq T^2(X) \cdots .$$

The subspace

$$\mathcal{N}^\infty(T) := \bigcup_{n=1}^{\infty} \ker T^n$$

is called the *hyper-kernel* of T, while

$$T^\infty(X) := \bigcap_{n=1}^{\infty} T^n(X)$$

is called the *hyper-range* of T. Note that both $\mathcal{N}^\infty(T)$ and $T^\infty(X)$ are T-invariant linear subspace of T, i.e.

$$T(\mathcal{N}^\infty(T)) \subseteq \mathcal{N}^\infty(T) \quad \text{and} \quad T(T^\infty(X)) \subseteq T^\infty(X).$$

The following elementary lemma will be useful in the sequel.

Lemma 1.12. *Let X be a vector space and T a linear operator on X. If p_1 and p_2 are relatively prime polynomials then there exist polynomials q_1 and q_2 such that $p_1(T)q_1(T) + p_2(T)q_2(T) = I$.*

Proof If p_1 and q_1 are relatively prime polynomials then there are polynomials such that $p_1(\mu)q_1(\mu) + p_2(\mu)q_2(\mu) = 1$ for every $\mu \in \mathbb{C}$. ■

The next result establishes some basic properties of the hyper-kernel and the hyper-range of an operator.

Theorem 1.13. *Let X be a vector space and T a linear operator on X. Then we have:*

(i) $(\lambda I + T)(\mathcal{N}^\infty(T)) = \mathcal{N}^\infty(T)$ *for every* $\lambda \neq 0$*;*

(ii) $\mathcal{N}^\infty(\lambda I + T) \subseteq (\mu I + T)^\infty(X)$ *for every* $\lambda \neq \mu$*.*

Proof (i) IIt is evident that the equality will be proved if we show that $(\lambda I + T)(\ker T^n) = \ker T^n$ for every $n \in \mathbb{N}$ and $\lambda \neq 0$. Clearly, $(\lambda I + T)(\ker T^n) \subseteq \ker T^n$ holds for all $n \in \mathbb{N}$. By Lemma 1.12 there exist polynomials p and q such that

$$(\lambda I + T)p(T) + q(T)T^n = I.$$

If $x \in \ker T^n$ then $(\lambda I + T)p(T)x = x$ and since $p(T)x \in \ker T^n$ this implies $\ker T^n \subseteq (\lambda I + T)(\ker T^n)$.

(ii) Put $S := \lambda I + T$ and write

$$\mu I + T = (\mu - \lambda)I + \lambda I + T = (\mu - \lambda)I + S.$$

By assumption $\mu - \lambda \neq 0$, so by part (i) we obtain that

$$(\mu I + T)(\mathcal{N}^\infty(\lambda I + T)) = ((\mu - \lambda)I + S)(\mathcal{N}^\infty(S)) = \mathcal{N}^\infty(\lambda I + T).$$

¿From this it easily follows that $(\mu I + T)^n(\mathcal{N}^\infty(\lambda I + T)) = \mathcal{N}^\infty(\lambda I + T)$ for all $n \in \mathbb{N}$, and consequently $\mathcal{N}^\infty(\lambda I + T) \subseteq (\mu I + T)^\infty(X)$. ■

Lemma 1.14. *For every linear operator T on a vector space X we have*

$$T^m(\ker T^{m+n}) = T^m(X) \cap \ker T^n \quad \textit{for all } m, n \in \mathbb{N}.$$

Proof If $x \in \ker T^{m+n}$ then $T^m x \in T^m(X)$ and $T^n(T^m x) = 0$, so that $T^m(\ker T^{m+n}) \subseteq T^m(X) \cap \ker T^n$.

Conversely, if $y \in T^m(X) \cap \ker T^n$ then $y = T^m(x)$ and $x \in \ker T^{m+n}$, so the opposite inclusion is verified. ■

In the next result we give some useful connections between the kernels and the ranges of the iterates T^n of an operator T on a vector space X.

Theorem 1.15. *For a linear operator T on a vector space X the following statements are equivalent:*

(i) $\ker T \subseteq T^m(X)$ *for each* $m \in \mathbb{N}$*;*

(ii) $\ker T^n \subseteq T(X)$ *for each* $n \in \mathbb{N}$*;*

(iii) $\ker T^n \subset T^m(X)$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$*;*

(iv) $\ker T^n = T^m(\ker T^{m+n})$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$*.*

Proof The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (i) If we apply the inclusion (ii) to the operator T^m we then obtain $\ker T^{mn} \subseteq T^m(X)$ and consequently $\ker T \subseteq T^m(X)$, since $\ker T \subseteq \ker T^{mn}$.

(i) $\Rightarrow$ (iv) If we apply the inclusion (i) to the operator T^n we obtain $\ker T^n \subseteq (T^n)^m(X) \subseteq T^m(X)$. By Lemma 1.14 we then have

$$T^m(\ker T^{m+n}) = T^m(X) \cap \ker T^n = \ker T^n,$$

so the proof is complete. ■

Corollary 1.16. *Let T be a linear operator on a vector space X. Then the statements of Theorem 1.15 are equivalent to each of the following inclusions:*

(i) $\ker T \subseteq T^\infty(X)$*;*

(ii) $\mathcal{N}^\infty(T) \subseteq T(X)$*;*

(iii) $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$*.*

We now introduce two important notions in Fredholm theory.

Definition 1.17. *Given a linear operator T on a vector space X, T is said to have* finite ascent *if* $\mathcal{N}^\infty(T) = \ker T^k$ *for some positive integer k. Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that* $\ker T^p = \ker T^{p+1}$. *The positive integer p is called the* ascent *of T. If there is no such integer we set $p(T) := \infty$. Analogously, T is said to have* finite descent *if $T^\infty(X) = T^k(X)$ for some k. The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the* descent *of T. If there is no such integer we set $q(T) := \infty$.*

Clearly $p(T) = 0$ if and only if T is injective and $q(T) = 0$ if and only if T is surjective. The following lemma establishes useful and simple characterizations of operators having finite ascent and finite descent.

Lemma 1.18. *Let T be a linear operator on a vector space X. For a positive natural m, the following assertions hold:*

(i) *$p(T) \le m < \infty$ if and only if for every $n \in \mathbb{N}$ we have $T^m(X) \cap \ker T^n = \{0\}$;*

(ii) *$q(T) \le m < \infty$ if and only if for every $n \in \mathbb{N}$ there exists a subspace $Y_n \subseteq \ker T^m$ such that $X = Y_n \oplus T^n(X)$.*

Proof (i) Suppose $p(T) \le m < \infty$ and n any natural number. Consider an element $y \in T^m(X) \cap \ker T^n$. Then there exists $x \in X$ such that $y = T^m x$ and $T^n y = 0$. From that we obtain $T^{m+n}x = T^n y = 0$ and therefore $x \in \ker T^{m+n} = \ker T^m$. Hence $y = T^m x = 0$.

Conversely, suppose $T^m(X) \cap \ker T^n = \{0\}$ for some natural m and let $x \in \ker T^{m+1}$. Then $T^m x \in \ker T$ and therefore

$$T^m x \in T^m(X) \cap \ker T \subseteq T^m(X) \cap \ker T^n = \{0\}.$$

Hence $x \in \ker T^m$. We have shown that $\ker T^{m+1} \subseteq \ker T^m$. Since the opposite inclusion is verified for all operators we conclude that $\ker T^m = \ker T^{m+1}$.

(ii) Let $q := q(T) \le m < \infty$ and Y be a complementary subspace to $T^n(X)$ in X. Let $\{x_j : j \in J\}$ be a basis of Y. For every element x_j of the basis there exists, since $T^q(Y) \subseteq T^q(X) = T^{q+n}(X)$, an element $y_j \in X$ such that $T^q x_j = T^{q+n} y_j$. Set $z_j := x_j - T^n y_j$. Then

$$T^q z_j = T^q x_j - T^{q+n} y_j = 0.$$

From this it follows that the linear subspace Y_n generated by the elements z_j is contained in $\ker T^q$ and *a fortiori* in $\ker T^m$. From the decomposition $X = Y \oplus T^n(X)$ we obtain for every $x \in X$ a representation of the form

$$x = \sum_{j \in J} \lambda_j x_j + T^n y = \sum_{j \in J} \lambda_j (z_j + T^n y_j) + T^n y = \sum_{j \in J} \lambda_j z_j + T^n z,$$

so $X = Y_n + T^n(X)$. We show that this sum is direct. Indeed, suppose that $x \in Y_n \cap T^n(X)$. Then $x = \sum_{j \in J} \mu_j z_j = T^n v$ for some $v \in X$, and therefore

$$\sum_{j \in J} \mu_j x_j = \sum_{j \in J} \mu_j T^n y_j + T^n v \in T^n(X).$$

¿From the decomposition $X = Y \oplus T^n(X)$ we then obtain that $\mu_j = 0$ for all $j \in J$ and hence $x = 0$. Therefore Y_n is a complement of $T^n(X)$ contained in $\ker T^m$. Conversely, if for $n \in \mathbb{N}$ the subspace $T^n(X)$ has a complement $Y_n \subseteq \ker T^m$ then

$$T^m(X) = T^m(Y_n) + T^{m+n}(X) = T^{m+n}(X),$$

and therefore $q(T) \le m$. ■

Theorem 1.19. *If both $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$.*

Proof Set $p := p(T)$ and $q := q(T)$. Assume first that $p \le q$, so that the inclusion $T^q(X) \subseteq T^p(X)$ holds. Obviously we may assume $q > 0$. From part (ii) of Lemma 1.18 we have $X = \ker T^q + T^q(X)$, so every element $y := T^p(x) \in T^p(X)$ admits the decomposition $y = z + T^q w$, with $z \in \ker T^q$. From $z = T^p x - T^q w \in T^p(X)$ we then obtain that $z \in \ker T^q \cap T^p(X)$ and hence the last intersection is $\{0\}$ by part (i) of Lemma 1.18. Therefore $y = T^q w \in T^q(X)$ and this shows the equality $T^p(X) = T^q(X)$, from whence we obtain $p \ge q$, so that $p = q$.

Assume now that $q \le p$ and $p > 0$, so that $\ker T^q \subseteq \ker T^p$. From part (ii) of Lemma 1.18 we have $X = \ker T^q + T^p(X)$, so that an arbitrary element x of $\ker T^p$ admits the representation $x = u + T^p v$, with $u \in \ker T^q$. From $T^p x = T^p u = 0$ it then follows that $T^{2p} v = 0$, so that $v \in \ker T^{2p} = \ker T^p$. Hence $T^p v = 0$ and consequently $x = u \in \ker T^q$. This shows that $\ker T^q = \ker T^p$, hence $q \ge p$. Therefore $p = q$. ■

In the sequel, for every bounded operator $T \in L(X, Y)$, we shall denote by $\alpha(T)$ the *nullity* of T, defined as $\alpha(T) := \dim \ker T$, while the *deficiency* $\beta(T)$ of T is defined $\beta(T) := \operatorname{codim} T(X) = \dim Y/T(X)$.

Let $\Delta(X)$ denote the set of all linear operators defined on a vector space X for which the nullity $\alpha(T)$ and the deficiency $\beta(T)$ are both finite. For every $T \in \Delta(X)$, the *index* of T , is defined by

$$\text{ind } T := \alpha(T) - \beta(T).$$

For the following result see Theorem 23.1 of [**68**], or also the proof of next Theorem 1.47.

Theorem 1.20. (index theorem) *Let X be a vector space, $T, S \in \Delta(X)$. Then $ST \in \Delta(X)$. Moreover,* $\text{ind}\, TS = \text{ind}\, T + \text{ind}\, S$.

The next theorem we establish the basic relationships between the quantities $\alpha(T)$, $\beta(T)$, $p(T)$ and $q(T)$.

Theorem 1.21. *If T is a linear operator on a vector space X then the following properties hold:*

(i) *If $p(T) < \infty$ then $\alpha(T) \leq \beta(T)$;*

(ii) *If $q(T) < \infty$ then $\beta(T) \leq \alpha(T)$;*

(iii) *If $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinite);*

(iv) *If $\alpha(T) = \beta(T) < \infty$ and if either $p(T)$ or $q(T)$ is finite then $p(T) = q(T)$.*

Proof (i) Let $p := p(T) < \infty$. Obviously if $\beta(T) = \infty$ there is nothing to prove. Assume that $\beta(T) < \infty$. It is easy to check that also $\alpha(T^n)$ is finite. By Lemma 1.18, part (i), we have $\ker T \cap T^p(X) = \{0\}$ and this implies that $\alpha(T) < \infty$. From the index theorem we obtain for all $n \geq p$ the following equality:

$$n \cdot \text{ind } T = \text{ind } T^n = \alpha(T^p) - \beta(T^n).$$

Now suppose that $q := q(T) < \infty$. For all integers $n \geq \max\{p, q\}$ the quantity $n \cdot \text{ind } T = \alpha(T^p) - \beta(T^p)$ is then constant, so that $\text{ind } T = 0$, $\alpha(T) = \beta(T)$. Consider the other case $q = \infty$. Then $\beta(T^n) \to 0$ as $n \to \infty$, so $n \cdot \text{ind } T$ eventually becomes negative, and hence $\text{ind } T < 0$. Therefore in this case we have $\alpha(T) < \beta(T)$.

(ii) Let $q := q(T) < \infty$. Also here we can assume that $\alpha(T) < \infty$, otherwise there is nothing to prove. Consequently, as is easy to check, also $\beta(T^n) < \infty$ and by part (ii) of Lemma 1.18 $X = Y \oplus T(X)$ with $Y \subseteq \ker T^q$. From this it follows that $\beta(T) = \dim\, Y \leq \alpha(T^q) < \infty$. If we use, with appropriate changes, the index argument used in the proof of part (i) then we obtain that $\beta(T) = \alpha(T)$ if $p(T) < \infty$, and $\beta(T) < \alpha(T)$ if $p(T) = \infty$.

(iii) It is clear from part (i) and part (ii).

(iv) This is an immediate consequence of the equality $\alpha(T^n)-\beta(T^n) = \text{ind } T^n = n\cdot \text{ind } T = 0$, valid for every $n \in \mathbb{N}$. ∎

3. Algebraic and analytic core

In this section we shall introduce some important T-invariant subspaces. The first one has been introduced by Saphar [**96**]

Definition 1.22. *Given a linear operator T defined on a vector space X, the* algebraic core $C(T)$ *of T is defined to be the greatest linear subspace M such that $T(M) = M$.*

It is easy to prove that $C(T)$ is the set of all $x \in X$ such that there exists a sequence $(x_n)_{n=0,1,\dots}$ such that $x_0 = x$, $Tx_{n+1} = x_n$ for all $n = 0, 1, 2, \dots$.

Trivially, if $T \in L(X)$ is surjective then $C(T) = X$. Clearly, for every linear operator T we have $C(T) = T^n(C(T)) \subseteq T^n(X)$ for all $n \in \mathbb{N}$. From that it follows that $C(T) \subseteq T^\infty(X)$. The next result shows that under certain purely algebraic conditions the algebraic core and the hyper-range of an operator coincide.

Lemma 1.23. *Let T be a linear operator on a vector space X. Suppose that there exists $m \in \mathbb{N}$ such that*

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X) \quad \textit{for all integers } k \geq 0.$$

Then $C(T) = T^\infty(X)$.

Proof We have only to prove that $T^\infty(X) \subseteq C(T)$. We show that $T(T^\infty(X)) = T^\infty(X)$. Evidently the inclusion $T(T^\infty(X)) \subseteq T^\infty(X)$ holds for every linear operator, so we need only to prove the opposite inclusion.

Let $D := \ker T \cap T^m(X)$. Obviously we have

$$D = \ker T \cap T^m(X) = \ker T \cap T^\infty(X).$$

Let us now consider an element $y \in T^\infty(X)$. Then $y \in T^n(X)$ for each $n \in \mathbb{N}$, so there exists $x_k \in X$ such that $y = T^{m+k}x_k$ for every $k \in \mathbb{N}$. If we set

$$z_k := T^m x_1 - T^{m+k-1}x_k \quad (k \in \mathbb{N}),$$

then $z_k \in T^m(X)$ and since

$$Tz_k = T^{m+1}x_1 - T^{m+k}x_k = y - y = 0$$

we also have $z_k \in \ker T$. Thus $z_k \in D$, and from the inclusion

$$D = \ker T \cap T^{m+k}(X) \subseteq \ker T \cap T^{m+k-1}(X)$$

it follows that $z_k \in T^{m+k-1}(X)$. This implies that

$$T^m x_1 = z_k + T^{m+k-1} x_k \in T^{m+k-1}(X)$$

for each $k \in \mathbb{N}$, and therefore $T^m x_1 \in T^\infty(X)$. Finally, from

$$T(T^m x_1) = T^{m+1} x_1 = y$$

we may conclude that $y \in T(T^\infty(X))$. Therefore $T^\infty(X) \subseteq T(T^\infty(X))$, so the proof is complete. ∎

Theorem 1.24. *Let T be a linear operator on a vector space X. Suppose that one of the following conditions holds:*

(i) $\alpha(T) < \infty$*;*

(ii) $\beta(T) < \infty$*;*

(iii) $\ker T \subseteq T^n(X)$ *for all* $n \in \mathbb{N}$*.*

Then $C(T) = T^\infty(X)$*.*

Proof (i) If $\ker T$ is finite-dimensional then there exists a positive integer m such that

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X)$$

for all integers $k \geq 0$. Hence it suffices to apply Lemma 1.23.

(ii) Suppose that $X = F \oplus T(X)$ with $\dim F < \infty$. Clearly, if we let $D_n := \ker T \cap T^n(X)$ then we have $D_n \supseteq D_{n+1}$ for all $n \in \mathbb{N}$. Suppose that there exist k distinct subspaces D_n. There is no loss of generality in assuming $D_j \neq D_{j+1}$ for $j = 1, 2, \dots k$. Then for every one of these j we can find an element $w_j \in X$ such that $T^j w_j \in D_j$ and $T^j w_j \notin D_{j+1}$. By means of the decomposition $X = F \oplus T(X)$ we also find $u_j \in F$ and $v_j \in T(X)$ such that $w_j = u_j + v_j$. We claim that the vectors $u_1, \cdots, u_k$ are linearly independent.

To see this let us suppose $\sum_{j=1}^k \lambda_j u_j = 0$. Then

$$\sum_{j=1}^k \lambda_j w_j = \sum_{j=1}^k \lambda_j v_j$$

and therefore from the equalities $T^k w_1 = \cdots = T^k w_{k-1} = 0$ we deduce that

$$T^k(\sum_{j=1}^{k} \lambda_j w_j) = \lambda_k T^k w_k = T^k(\sum_{j=1}^{k} \lambda_j v_j) \in T^k(T(X)) = T^{k+1}(X).$$

¿From $T^k w_k \in \ker T$ we obtain $\lambda_k T^k w_k \in D_{k+1}$, and since $T^k w_k \notin D_{k+1}$ this is possible only if $\lambda_k = 0$. Analogously we have $\lambda_{k-1} = \cdots = \lambda_1 = 0$, so the vectors $u_1, \dots, u_k$ are linearly independent. From this it follows that k is smaller than or equal to the dimension of F . But then for a sufficiently large m we obtain that

$$\ker T \cap T^m(X) = \ker T \cap T^{m+j}(X)$$

for all integers $j \geq 0$. So we are again in the situation of Lemma 1.23.

(iii) Obviously, if $\ker T \subseteq T^n(X)$ for all $n \in \mathbb{N}$, then

$$\ker T \cap T^n(X) = \ker T \cap T^{n+k}(X) = \ker T$$

for all integers $k \geq 0$. Hence also in this case we can apply Lemma 1.23. ■

The finiteness of $p(T)$ or $q(T)$ has also some remarkable consequences on $T|\, T^\infty(X)$, the restriction of T on $T^\infty(X)$.

Theorem 1.25. *Let T be a linear operator on the vector space X. We have:*

(i) *If either $p(T)$ or $q(T)$ is finite then $T\ |T^\infty(X)$ is surjective. To be precise $T^\infty(X) = C(T)$.*

(ii) *If either $\alpha(T) < \infty$ or $\beta(T) < \infty$ then*

$$p(T) < \infty \Leftrightarrow T|T^\infty(X) \text{ is injective.}$$

Proof (i) The assertion follows immediately from Lemma 1.23 because if $p = p(T) < \infty$ then by Lemma 1.18

$$\ker T \cap T^p(X) = \ker T \cap T^{p+k}(X) \quad \text{for all integers } k \geq 0;$$

whilst if $q = q(T) < \infty$ then

$$\ker T \cap T^q(X) = \ker T \cap T^{q+k}(X) \quad \text{for all integers } k \geq 0.$$

(ii) Assume that $p(T) < \infty$. We have $C(T) = T^\infty(X)$ and hence $T(T^\infty(X)) = T^\infty(X)$. Let $\widetilde{T} := T|T^\infty(X)$. Then $\widetilde{T}$ is surjective, thus $q(\widetilde{T}) = 0$. From our assumption and from the equality $\ker \widetilde{T}^n = \ker T^n \cap T^\infty(X)$ we also obtain $p(\widetilde{T}) < \infty$. From Theorem 1.19 we then conclude that $p(\widetilde{T}) = q(\widetilde{T}) = 0$, and therefore the restriction $\widetilde{T}$ is injective.

Conversely, if $\widetilde{T}$ is injective then $\ker T \cap T^\infty(X) = \{0\}$. By assumption $\alpha(T) < \infty$ or $\beta(T) < \infty$, and this implies (see the proof of Theorem 1.24) that $\ker T \cap T^n(X) = \{0\}$ for some positive integer n. By Lemma 1.18 it then follows that $p(T) < \infty$. ■

The finiteness of the ascent and the descent of a linear operator T is related to a certain decomposition of X.

Theorem 1.26. *Suppose that T is a linear operator on a vector space X. If $p = p(T) = q(T) < \infty$ then we have the decomposition*

$$X = T^p(X) \oplus \ker T^p.$$

Conversely, if for a natural number m we have the decomposition $X = T^m(X) \oplus \ker T^m$ then $p(T) = q(T) \leq m$. In this case $T|T^p(X)$ is bijective.

Proof If $p < \infty$ and we may assume that $p > 0$, then the decomposition $X = T^p(X) \oplus \ker T^p$ immediately follows from Lemma 1.18. Conversely, if $X = T^m(X) \oplus \ker T^m$ for some $m \in \mathbb{N}$ then $p(T), q(T) \leq m$, again by Lemma 1.18, and hence $p(T) = q(T) < \infty$ by Theorem 1.19.

To verify the last assertion observe that $T^\infty(X) = T^p(X)$, so from Theorem 1.25 $\widetilde{T} := T|T^p(X)$ is onto. On the other hand, $\ker \widetilde{T} \subseteq \ker T \subseteq \ker T^p$, but also $\ker \widetilde{T} \subseteq T^p(X)$, so the decomposition $X = T^p(X) \oplus \ker T^p$ entails that $\ker \widetilde{T} = \{0\}$. ■

The following subspace has been introduced by Vrbová [**103**] and studied in several papers by Mbekhta ([**78**], [**80**], [**79**]. This subspace is, in a certain sense, the analytic counterpart of the algebraic core $C(T)$.

Definition 1.27. *Let X be a Banach space and $T \in L(X)$. The* analytical core *of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and a constant $\delta > 0$ such that:*

(1) $x = u_0$, *and* $Tu_{n+1} = u_n$ *for every* $n \in \mathbb{Z}_+$;

(2) $\|u_n\| \leq \delta^n \|x\|$ *for every* $n \in \mathbb{Z}_+$.

In the following theorem we collect some elementary properties of $K(T)$.

Theorem 1.28. *Let $T \in L(X)$, X a Banach space. Then:*

(i) *$K(T)$ is a linear subspace of X;*

(ii) $T(K(T)) = K(T)$;

(iii) $K(T) \subseteq C(T)$.

Proof (i) It is evident that if $x \in K(T)$ then $\lambda x \in K(T)$ for every $\lambda \in \mathbb{C}$. We show that if $x, y \in K(T)$ then $x+y \in K(T)$. If $x \in K(T)$ there exists $\delta_1 > 0$ and a sequence $(u_n) \subset X$ satisfying the condition (1) and which is such that $\|u_n\| \leq {\delta_1}^n \|x\|$ for all $n \in \mathbb{Z}_+$. Analogously, since $y \in K(T)$ there exists $\delta_2 > 0$ and a sequence $(v_n) \subset X$ satisfying the condition (1) of the definition of $K(T)$ and such that $\|v_n\| \leq \delta_2^n \|y\|$ for every $n \in \mathbb{N}$.

Let $\delta : = \max\{\delta_1, \delta_2\}$. We have

$$\|u_n + v_n\| \leq \|u_n\| + \|v_n\| \leq \delta_1^n \|x\| + \delta_2^n \|y\| \leq \delta^n (\|x\| + \|y\|).$$

Trivially, if $x+y = 0$ there is nothing to prove since $0 \in K(T)$. Suppose then $x + y \neq 0$ and set

$$\mu := \frac{\|x\| + \|y\|}{\|x + y\|}$$

Clearly $\mu \geq 1$, so $\mu \leq \mu^n$ and therefore

$$\|u_n + v_n\| \leq (\delta)^n \mu \|x + y\| \leq (\delta\mu)^n \|x + y\| \quad \text{for all } n \in \mathbb{Z}_+,$$

which shows that also the property (2) of the definition of $K(T)$ is verified for every sum $x + y$, with x, $y \in K(T)$. Hence $x + y \in K(T)$, and consequently $K(T)$ is a linear subspace of X.

The proof (ii) is rather simple, while (iii) is a trivial consequence of (ii) and the definition of $C(T)$. ∎

Observe that in general neither $K(T)$ nor $C(T)$ are closed. The next result shows that under the assumption that $C(T)$ is closed then these two subspaces coincide.

Theorem 1.29. *Suppose X a Banach space and $T \in L(X)$.*

(i) *If F is a closed subspace of X such that $T(F) = F$ then $F \subseteq K(T)$.*

(ii) *If $C(T)$ is closed then $C(T) = K(T)$.*

Proof (i): Let $T_0 : F \to F$ denote the restriction of T on F. By assumption F is a Banach space and $T(F) = F$, so, by the open mapping theorem, T_0 is open. This means that there exists a constant $\delta > 0$ with the property that for every $x \in F$ there is $u \in F$ such that $Tu = x$ and $\|u\| \leq \delta \|x\|$.

Now, if $x \in F$, define $u_0 := x$ and consider an element $u_1 \in F$ such that

$$Tu_1 = u_0 \quad \text{and} \quad \|u_1\| \leq \delta \|u_0\|.$$

By repeating this procedure, for every $n \in \mathbb{N}$ we find an element $u_n \in F$ such that

$$Tu_n = u_{n-1} \quad \text{and} \quad \|u_n\| \leq \delta \|u_{n-1}\|.$$

¿From the last inequality we obtain the estimate

$$\|u_n\| \leq \delta^n \|u_0\| = \delta^n \|x\| \quad \text{for every} \quad n \in \mathbb{N},$$

so $x \in K(T)$. Hence $F \subseteq K(T)$.

(ii) Suppose that $C(T)$ is closed. Since $C(T) = T(C(T))$ the first part of the theorem shows that $C(T) \subseteq K(T)$, and hence, since the reverse inclusion is always true, $C(T) = K(T)$. ■

4. Compact operators

In this section we reassume some of the basic properties of compact linear operators. Before of defining this class of operators, we shall consider an important integral equation. Fix a continuous function $y(s)$ on the interval $[a, b]$, and let k be a continuous function on $[a, b] \times [a, b]$. The *Fredholm integral equation* (of second kind) then is the functional equation defined by

$$\lambda x(s) - \int_a^b k(s,t)x(t)\mathrm{d}t = y(s), \quad \lambda \neq 0. \tag{6}$$

Our aim is to find a continuous function $x(s)$ such that (6) is satisfied. Let $C[a, b]$ denote the vector space of all continuous complex-valued functions on $[a, b]$. With respect to the norm (of uniform convergence)

$$\|x\| := \max_{t \in [a,b]} |x(t)|, \quad x \in C[a, b]$$

then $C[a, b]$ is a Banach space. If we define $K : C[a, b] \to C[a, b]$ by

$$(Kx)(s) := \int_a^b k(s,t)x(t)\mathrm{d}t, \tag{7}$$

it is easily seen that K is a linear bounded operator and the equation (6) may be written

$$(\lambda I - K)x = y, \quad y \in C[a, b].$$

Clearly, the equation (6) has a solution if y belongs to the range of $\lambda I - K$, and this solution will be unique if $\lambda I - K$ is injective. Now, according the classical Weierstrass approximation theorem, there exists a sequence of polynomials

$$p_n(s,t) := \sum_{i,k} \gamma_{ik} s^i t^k \quad s,t \in [a,b],$$

such that $p_n(s,t)$ converges uniformly on $[a,b] \times [a,b]$, i.e.

$$\max_{a \le s,t \le b} |p_n(s,t) - k(s,t)| \to 0 \quad \text{as } n \to \infty.$$

Let us consider the integral operators P_n on $C[a,b]$ defined by

$$(P_n x)(s) := \int_a^b p_n(s,t)x(t)\mathrm{d}t.$$

Then

$$\|(P_n - K)x\| \le (b-a)\big(\max_{s,t\in[a,b]} |p_n(s,t) - k(s,t)|\,\big)\,\|x\|,$$

and from this it easily follows that the sequence of operators (P_n) converges in $L(C[a,b])$ to the integral operator K. Now, every polynomial $p(s,t)$ may be written in the form

$$p(s,t) = \sum_{i=1}^m g_i(s)h_i(t), \quad g_i, h_i \in C[a,b]$$

and from this it is easy to see that the corresponding integral operator P, defined by

$$(Px)(s) := \int_a^b p(s,t)x(t)\mathrm{d}t,$$

satisfies the equality

$$Px = \sum_{i=1}^m \mu_i g_i, \quad \text{with} \quad g_i \in C[a,b],$$

where the scalars $\mu_i \in \mathbb{C}$ are defined by $\mu_i := \int_a^b h_i(t)x(t)\mathrm{d}t$. This shows that the range of P is contained in the linear subspace of $C[a,b]$ generated by the set $\{g_1, g_2, \dots g_n\}$. The operator P is a finite-dimensional operator, where a an operator $T \in L(X,Y)$ is said to be *finite-dimensional* if its range $T(X)$ is finite-dimensional. The integral operator K is then the limit, in the norm topology, of a sequence (P_n) of finite-dimensional operators.

Definition 1.30. *A bounded operator T from a normed space X into a normed space Y is said to be* compact *if for every bounded sequence (x_n) of elements of X the corresponding sequence (Tx_n) contains a convergent subsequence. This is equivalent to saying that the closure of $T(B_X)$, B_X the closed unit ball of X, is a compact subset of Y.*

Denote by $\mathcal{F}(X,Y)$ the set of all continuous finite-dimensional operators, and by $\mathcal{K}(X,Y)$ the set of all compact operators. In the sequel we list some basic properties of these sets (see [**68**, §13]).

Theorem 1.31. *Let X, Y and Z be Banach spaces. Then*

(i) *$\mathcal{F}(X,Y)$ and $\mathcal{K}(X,Y)$ are linear subspaces of $L(X,Y)$. Moreover, $\mathcal{F}(X,Y) \subseteq \mathcal{K}(X,Y)$.*

(ii) *If $T \in \mathcal{F}(X,Y)$, $S \in L(Y,Z)$, $U \in L(Z,X)$ then $ST \in \mathcal{F}(X,Z)$ and $TU \in \mathcal{F}(Z,Y)$. Analogous statements hold for $T \in \mathcal{K}(X,Y)$.*

(iii) *If (T_n) is a sequence of $\mathcal{K}(X,Y)$ which converges to T then $T \in \mathcal{K}(X,Y)$. Consequently, $\mathcal{K}(X,Y)$ is a closed subspace of $L(X,Y)$.*

Note that by Theorem 1.31, part (ii), the integral operator K defined in (7) is compact. A famous counter-example of Enflo [**53**] shows that not every compact operator is the limit of finite-dimensional operators.

Note that $\mathcal{K}(X,Y)$ may be coincide with $L(X,Y)$. This for instance is the case where $X = \ell^q$ or $X = c_0$, $Y = \ell^p$, with $1 \leq p < q < \infty$, see [**74**, 2.c.3].

Let now consider the case $X = Y$ and set $\mathcal{F}(X) := \mathcal{F}(X,X)$, $\mathcal{K}(X) := \mathcal{K}(X,X)$. Recall that a subset J of a Banach algebra $\mathcal{A}$ is said to be a (two-sided) *ideal* if J is a linear subspace of $\mathcal{A}$ and for every $x \in J$, $a \in \mathcal{A}$, the products xa and ax lie in J. From Theorem 1.31 we then deduce that $\mathcal{F}(X)$ as well as $\mathcal{K}(X)$ are ideals of the Banach algebra $L(X)$. Note that if X is infinite-dimensional then $I \notin K(X)$, otherwise any bounded sequence would contain a convergent subsequence and by Bolzano-Weiestrass theorem this is not possible. It is known that we can define on the quotient algebra $L(X)/\mathcal{K}(X)$ the *quotient norm*:

$$\|\hat{T}\| := \inf_{T \in \hat{T}} \|T\|, \quad \text{where } \hat{T} := T + \mathcal{K}(X).$$

Since $\mathcal{K}(X)$ is closed then the quotient algebra $\hat{\mathcal{L}} := L(X)/\mathcal{K}(X)$, with respect to the quotient norm above defined, is a Banach algebra, known in literature as the *Calkin algebra*. Also $L(X)/\mathcal{F}(X)$ is an algebra, but in general is not a Banach algebra.

Compactness is preserved by duality:

Theorem 1.32. (Schauder theorem) *If $T \in L(X, Y)$, X and Y Banach spaces, then T is compact if and only if T^* is compact.*

Proof See [**68**, §42]. ∎

In the sequel we will need the following important lemma.

Lemma 1.33. (Riesz lemma) *Let Y be a proper closed subspace of a normed space X. Then for every $0 < \delta < 1$ there exists a vector $x_\delta \in X$ such that $\|x_\delta\| = 1$ and*

$$\|y - x_\delta\| \geq \delta \quad \textit{for all } y \in Y.$$

Proof Let $y \in X$ such that $y \notin Y$. Set $\rho := \inf_{x \in Y} \|x - y\|$ and let (x_n) be a sequence such that $\|x_n - y\| \to \rho$ as $n \to \infty$. Since Y is closed we have $\rho > 0$. Now, if $0 < \delta < 1$ then $\rho/\delta > \rho$, hence there exists $z \in Y$ such that $0 < \|z - y\| \leq \rho/\delta$. Setting $\gamma := 1/\|z - y\|$ and $x_\delta := \gamma(y - z)$ we then obtain $\|x_\delta\| = 1$. Since $(1/\gamma)x + z \in Y$ and $\gamma \geq \delta/\rho$ it then follows that

$$\|x - x_\delta\| = \gamma\|(\frac{1}{\gamma}x + z) - y\| \geq \frac{\delta}{\rho} \cdot \rho = \delta,$$

as desired. ∎

Riesz Lemma has many important consequences. One of the most important is that the Bolzano-Weiestrass theorem holds in normed spaces X, see [**68**, Theorem 101].

Theorem 1.34. (Bolzano-Weiestrass theorem) *Suppose that X is a normed vector space. Then every bounded sequence contains a convergent subsequence precisely when the space X is finite-dimensional.*

We now establish some important properties of compact endomorphisms.

Theorem 1.35. *Let $T \in \mathcal{K}(X)$, X Banach space. Then $\alpha(\lambda I - T) < \infty$ and $(\lambda I - T)(X)$ is closed for all $\lambda \neq 0$.*

Proof We can suppose $\lambda = 1$ since $\lambda I - T = \lambda(I - \frac{1}{\lambda}T)$ and $\frac{1}{\lambda}T$ is compact. If (x_n) is a bounded sequence in $\ker(\lambda I - T)$ we have $Tx_n = x_n$. Since T is compact then there exists a convergent subsequence of $(Tx_n) = (x_n)$, so from Bolzano-Weiestrass theorem we deduce that $\ker(I - T)$ is finite-dimensional.

To show that $(I-T)(X)$ is closed, set $S := I - T$. We show that $y_n := Sx_n \to y$ implies $y \in S(X)$. Let

$$\lambda_n := \inf_{u \in \ker S} \|x_n - u\|.$$

Then for every n there exists $u_n \in \ker S$ such that $\|x_n - u_n\| \leq 2\lambda_n$ and if we set $v_n := x_n - u_n$ then $y_n = Sv_n$ and $\|v_n\| \leq 2\lambda_n$. We claim that the sequence (v_n) is bounded. Suppose that (v_n) is unbounded. Then it contains a subsequence, which will be denoted again by (v_n), such that $\|v_n\| \to \infty$. If we set $w_n := \frac{v_n}{\|v_n\|}$, it easily follows that $Sw_n \to 0$. Since $\|w_n\| = 1$, the compactness of T implies the existence of a convergent subsequence of (Tw_n). Let (Tw_{n_j}) be such a sequence and say that $Tw_{n_j} \to z$. Clearly,

$$w_{n_j} = (I-T)w_{n_j} + Tw_{n_j} = Sw_{n_j} + Tw_{n_j} \to z.$$

Consequently, $Sz = \lim Sw_{n_j} = 0$, thus $z \in \ker S$. An easy estimate yields

$$\|w_n - z\| = \|\frac{x_n - u_n - z}{\|v_n\|}\| = \frac{1}{\|v_n\|}\,\|x_n - (u_n + \|v_n\|z)\| \geq \frac{\lambda_n}{\|v_n\|},$$

and this is impossible, since $w_{n_j} \to z$. Thus, (v_n) is bounded and since T is compact then (Tv_n) contains a convergent subsequence (Tv_{n_j}). From $v_{n_j} = Sv_{n_j} + Tv_{n_j} = y_{n_j} + Tv_{n_j}$ we see that (v_{n_j}) converges to some $v \in X$, so that

$$y = \lim y_n = \lim y_{n_j} = \lim Sv_{n_j} = Sv \in S(X),$$

thus $S(X) = (I-T)(X)$ is closed. ■

Theorem 1.36. *If $T \in \mathcal{K}(X)$, X Banach space, then $p(\lambda I - T) = q(\lambda I - T) < \infty$ for all $\lambda \neq 0$.*

Proof Also here we can suppose $\lambda = 1$. Set

$$S := I - T, \quad Y_n := \ker S^n, \quad Z_n := S^n(X).$$

¿From the classical Newton binomial formula and taking into account that $\mathcal{K}(X)$ is an ideal we see that

$$S^n = (I-T)^n = I - [nT - \frac{1}{2}n(n-1)T^2 + \cdots (-1)^n T^n] = I - T_n,$$

where $T_n \in \mathcal{K}(X)$. Therefore, by Theorem 1.35, all Y_n are finite-dimensional and all Z_n are closed.

Suppose now that $p(S) = \infty$ Then Y_{n-1} is a closed proper subspace of Y_n, so by Riesz Lemma there is $x_n \in Y_n$ such that $\|x_n\| = 1$ and $\|x - x_n\| \geq \frac{1}{2}$ for all $x \in Y_{n-1}$. Since

$$Tx_n - Tx_m = x_n - (x_n - Sx_m + Sx_n)$$

and $x_n - Sx_m + Sx_n \in Y_{n-1}$ for all $m = 1, \dots, n-1$ then for these m we have

$$\|Tx_n - Tx_m\| \geq \frac{1}{2} \quad \text{for all } m = 1, \dots, n-1.$$

Hence (Tx_n) does not admit any convergent subsequence, contradicting the compactness of T. Thus $p(S) = p(I - T) < \infty$.

To conclude the proof, suppose that $q(S) = \infty$. Since Z_n is closed for each n there exists $x_n \in Z_n$ such that $\|x_n\| = 1$ and $\|x - x_n\| \geq \frac{1}{2}$ for all $x \in Z_{n+1}$. Again, since $x_n - Sx_m + Sx_n \in Z_{n+1}$ for all $m > n$, it then follows that

$$\|Tx_n - Tx_m\| \geq \frac{1}{2} \quad \text{for all } m > n,$$

and this, once more, contradicts the compactness of T. Therefore $q(S) = q(I - T) < \infty$. ∎

Combining Theorem 1.36, Theorem 1.35 with Theorem 1.21 we then obtain:

Corollary 1.37. *If $T \in \mathcal{K}(X)$, X a Banach space, then $\beta(\lambda I - T) < \infty$ and ind$(\lambda I - T) = 0$ for all $\lambda \neq 0$.*

Theorem 1.38. *If X and Y are Banach spaces and $T \in L(X, Y)$ is a compact operator having closed range then T is finite-dimensional.*

Proof Suppose first that T is compact and bounded below, namely T has closed range and is injective. Let (y_n) be a bounded sequence of $T(X)$ and let (x_n) be a sequence of X for which $Tx_n = y_n$ for all $n \in \mathbb{N}$. Since T is bounded below there exists a $\delta > 0$ such that $\|y_n\| = \|Kx_n\| \geq \delta \|x_n\|$ for all $n \in \mathbb{N}$, so (x_n) is bounded. The compactness of T then implies that there exists a subsequence (x_{n_k}) of (x_n) such that $Tx_{n_k} = y_{n_k}$ converges as $k \to \infty$. Hence every bounded sequence of $T(X)$ contains a convergent subsequence and by the Stone-Weiestrass theorem this implies that $T(X)$ is finite-dimensional.

Assume now the more general case that $T \in \mathcal{K}(X, Y)$ has closed range. If $T_0 : X \to T(X)$ is defined by $T_0 x := Tx$ for all $x \in X$, then T_0 is a compact operator from X onto $T(X)$. Therefore the dual ${T_0}^* : T(X)^* \to X^*$ is a compact operator by Schauder theorem. Moreover, T_0 is onto, so ${T_0}^*$ is bounded below by Theorem 1.6. The first part of the proof then gives that ${T_0}^*$ is finite-dimensional, and hence $T(X)^*$ is finite-dimensional because it is isomorphic to the range of ${T_0}^*$. From this it follows that also $T(X)$ is finite-dimensional. ∎

We recall in the sequel some basic definitions. A set Γ is said to be a *directed* if Γ is a partially ordered set such that, given λ_1, $\lambda_2 \in \Gamma$, there exists $\lambda \in \Gamma$ with $\lambda_k \leq \lambda$ $(k = 1, 2$. A *net* (x_α) in a topological space X is a mapping of a directed set Γ into X. Suppose now that X is a normed space. A net (x_λ) is said to be *weakly convergent* to x, in symbol $x_\lambda \rightharpoonup x$, if $f(x_\alpha) \to f(x)$ for all $f \in X^*$. Note that convergence in the sense of norm implies weak convergence.

The next characterization of compact operators will be needed in the sequel, for a proof see Müller [**84**, Theorem 15.5] .

Theorem 1.39. *If $T \in L(X, Y)$ then the following statements are equivalent:*

(i) *$T \in \mathcal{K}(X, Y)$;*

(ii) *If (x_λ) is a net, with $\|x_\lambda\| \leq 1$ for all λ, and $x_\lambda \rightharpoonup 0$ then $\|Tx_\lambda\|$ converges to 0;*

(iii) *For every $\varepsilon > 0$ there exists a finite-codimensional closed subspace M such that $\|T|M\| \leq \varepsilon$.*

5. Semi-Fredholm operators

We now introduce some important classes of operators.

Definition 1.40. *Given two Banach spaces X and Y, the set of all* upper semi-Fredholm operators *is defined by*

$$\Phi_+(X, Y) := \{T \in L(X, Y) : \alpha(T) < \infty \quad \textit{and } T(X) \textit{ is closed}\},$$

while the set of all lower semi-Fredholm operators *is defined by*

$$\Phi_-(X, Y) := \{T \in L(X, Y) : \beta(T) < \infty\}.$$

The set of all semi-Fredholm operators *is defined by*

$$\Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y).$$

The class $\Phi(X, Y)$ of all Fredholm operators *is defined by*

$$\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y).$$

At first glance the definitions of semi-Fredholm operators seems to be asymmetric, but this is not the case since the condition $\beta(T) < \infty$ entails by Corollary 1.8 that $T(X)$ is closed.

We shall set

$$\Phi_+(X) := \Phi_+(X, X) \quad \text{and} \quad \Phi_-(X) := \Phi_-(X, X),$$

while

$$\Phi(X) := \Phi(X, X) \quad \text{and} \quad \Phi_\pm(X) := \Phi_\pm(X, X).$$

If $T \in \Phi_\pm(X, Y)$ the *index* of T is defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$. Clearly, if T is bounded below then T is upper semi-Fredholm with index less or equal to 0, while any surjective operator is lower semi-Fredholm with index greater or equal to 0. Clearly, if $T \in \Phi_+(X, Y)$ is not Fredholm then $\operatorname{ind} T = -\infty$, while if $T \in \Phi_-(X, Y)$ is not Fredholm then $\operatorname{ind} T = \infty$.

Observe that in the case $X = Y$ the class $\Phi(X)$ is non-empty since the identity trivially is a Fredholm operator. This is a substantial difference from the case in which X and Y are different. In fact, if $T \in \Phi(X, Y)$ for some infinite-dimensional Banach spaces X and Y then there exist two subspaces M and N such that $X = \ker\, T \oplus M$ and $Y = T(X) \oplus N$, with M and $T(X)$ closed infinite-dimensional subspaces of X and Y, respectively. The restriction of T to M clearly has a bounded inverse, so the existence of a Fredholm operator from X into a different Banach space Y implies the existence of isomorphisms between some closed infinite-dimensional subspaces of X and Y. For this reason, for certain Banach spaces X, Y no bounded Fredholm operator from X to Y may exist, i.e., $\Phi(X, Y) = \varnothing$. This is for instance the case of $X := L^p[0, 1]$ and $Y = L^q[0, 1]$ with $0 < p < q < \infty$. Other examples may be found in Aiena [**1**, Chapter 7].

It can be proved that each finite-dimensional subspace as well as every closed finite-codimensional subspace is complemented, see Proposition 24.2 of Heuser [**68**]. Therefore, by Corollary 1.8 and Theorem 1.10 we have

Theorem 1.41. *Every Fredholm operator $T \in \Phi(X, Y)$ is relatively regular.*

Suppose now that $T \in L(X, Y)$ has closed range. Then, by using Theorem 1.3, we have

$$\begin{aligned} \alpha(T) &= \dim \ker T = \dim (\ker T)^* = \dim X^*/(\ker T)^\perp \\ &= \dim X^*/T^*(X^*) = \beta(T^*), \end{aligned}$$

and analogously

$$\begin{aligned}\beta(T) &= \dim Y/T(X) = \dim (Y/T(X))^* = \dim T(X)^\perp \\ &= \dim \ker T^* = \alpha(T^*).\end{aligned}$$

Hence,

$$T \in \Phi_+(X,Y) \Leftrightarrow T^* \in \Phi_-(Y^*,X^*),$$

and

$$T \in \Phi_-(X,Y) \Leftrightarrow T^* \in \Phi_+(Y^*,X^*).$$

Moreover, if $T \in \Phi_\pm(X,Y)$ then $\operatorname{ind} T^* = -\operatorname{ind} T$.

Theorem 1.42. *Suppose that X,Y and Z are Banach spaces.*

(i) *If $T \in \Phi_-(X,Y)$ and $S \in \Phi_-(Y,Z)$ then $ST \in \Phi_-(X,Z)$.*

(ii) *If $T \in \Phi_+(X,Y)$ and $S \in \Phi_+(Y,Z)$ then $ST \in \Phi_+(X,Z)$.*

(iii) *If $T \in \Phi(X,Y)$ and $S \in \Phi(Y,Z)$ then $ST \in \Phi(X,Z)$.*

In particular, if T belongs to one of the classes $\Phi_-(X,Y)$, $\Phi_+(X,Y)$, $\Phi_(X,Y)$ then T^n belongs to the same class for all $n \in \mathbb{N}$.

Proof (i) $T(X)$ and $S(Y)$ are complemented, since closed and finite-codimensional. Write $Y = T(X) \oplus M$ and $Z = S(Y) \oplus N$. Then

$$Z = N + S(T(X)) + S(M) = ST(X) + (N + S(M)),$$

where $N+S(M)$ is finite-dimensional. Hence $ST(X)$ has finite-codimension, i.e. $ST \in \Phi_-(X,Z)$.

The assertion (ii) is proved from (i) by duality. Indeed, T^* and S^* are lower semi-Fredholm and hence T^*S^* is lower semi-Fredholm. Therefore, ST is upper semi-Fredholm.

The assertion (iii) is obvious from (i) and (ii). ■

Corollary 1.43. *If $T \in \Phi_\pm(X)$ then $p(T) = q(T^*)$ and $q(T) = p(T^*)$.*

Proof If $T \in \mathit{\Phi}_\pm(X)$ then $T^n \in \mathit{\Phi}_\pm(X)$, and hence the range of T^n is closed for all n. Analogously, also T^{*n} has closed range, and therefore for every $n \in \mathbb{N}$,

$$\ker T^{n*} = T^n(X)^\perp, \quad \ker T^n =^\perp T^{n*}(X^*) =^\perp T^{*n}(X^*).$$

Obviously these equalities imply that $p(T^*) = q(T)$ and $p(T) = q(T^*)$. ■

¿From Theorem 1.35, Corollary 1.37 and Corollary 1.43 we also deduce the following important result:

Corollary 1.44. *If $T \in \mathcal{K}(X)$ then $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$. Moreover,*

$$\alpha(\lambda I - T) = \beta(\lambda I - T) = \alpha(\lambda I^* - T^*) = \beta(\lambda I^* - T^*) < \infty$$

and

$$p(\lambda I - T) = q(\lambda I - T) = p(\lambda I^* - T^*) = q(\lambda I^* - T^*) < \infty.$$

Corollary 1.44 has an important consequence for Fredholm integral equations (of second kind), the so-called *alternative criterion.*

Corollary 1.45. *If $T \in \Phi_\pm(X)$ then $K(T) = T^\infty(X)$ and $K(T)$ is closed.*

Proof If $T \in \Phi_\pm(X)$ then, by Theorem 1.42, $T^n \in \Phi_\pm(X)$ for all $n \in \mathbb{N}$, so the subspaces $T^n(X)$ are closed and hence $T^\infty(X)$ is closed. By Theorem 1.24 $C(T) = T^\infty(X)$, thus by Theorem 1.29 $C(T) = K(T)$. ■

Theorem 1.46. *Suppose that X,Y and Z are Banach spaces, $T \in L(X,Y)$, $S \in L(Y,Z)$.*

(i) *If $ST \in \Phi_-(X,Z)$ then $S \in \Phi_-(Y,Z)$.*

(ii) *If $ST \in \Phi_+(X,Z)$ then $T \in \Phi_+(X,Y)$.*

(iii) *If $ST \in \Phi(X,Z)$ then $T \in \Phi_+(X,Y)$ and $S \in \Phi_-(Y,Z)$.*

Proof i) Since $S(Y) \supseteq ST(X)$ then $\operatorname{codim} S(Y) \leq \operatorname{codim} ST(X)$.

ii) If $ST \in \Phi_+(X,Z)$ then $(ST)^* = T^*S^* \in \Phi_-(Z^*,X^*)$, so T^* is lower semi-Fredholm and hence T is upper semi-Fredholm.

(iii) It is obvious from (i) and (ii). ■

Theorem 1.47. *Let $T \in \Phi_-(X,Y)$ and $S \in \Phi_-(Y,X)$ (or $T \in \Phi_+(X,Y)$ and $S \in \Phi_+(Y,X)$), then $\mathit{ind}\,(ST) = \mathit{ind}\,S + \mathit{ind}\,T$.*

Proof Suppose first that T, S are Fredholm and let $M := T(X) \cap \ker S$. Clearly, M is a finite-dimensional subspace of $T(X)$, and hence complemented in $T(X)$. Write $T(X) = M \oplus N_1$ and $\ker S = M \oplus N_2$. From $N_2 \cap T(X) \subseteq \ker S \cap T(X)$ we obtain $N_2 \cap T(X) = \{0\}$. Choose a finite-dimensional subspace N_3 such that

$$Y = T(X) \oplus N_2 \oplus N_3 = M \oplus N_1 \oplus N_2 \oplus N_3.$$

Then

$$S(Y) = S(N_1) \oplus S(N_3) = S(N_1 \oplus N) \oplus S(N_3) = S(T(X) \oplus S(N_3),$$

from which we obtain $\beta(ST) = \beta(S) + \dim S(N_3) = \beta(S) + \dim N_3$. Moreover, $\alpha(S) = \dim M + \dim N_2$ and $\beta(T) = \dim N_2 + \dim N_3$. Let us consider the restriction $\hat{T} := T|\ker ST : \ker ST \to M$. Evidently, $\ker \hat{T} = \ker T$ and $\hat{T}$ is onto, so that $\dim M = \alpha(ST) - \alpha(T)$. From this we obtain

$$\begin{aligned} \operatorname{ind}(ST) &= \dim M + \alpha(T) - \beta(S) - \dim N_3 \\ &= \alpha(S) - \dim N_2 + \alpha(T) - \beta(S) - \beta(T) + \dim N_2 \\ &= \operatorname{ind} S + \operatorname{ind} T. \end{aligned}$$

Suppose now that $S \in \Phi_-(X, Y)$ but T not Fredholm. Then $\alpha(ST) \geq \alpha(T) = \infty$ and hence

$$\operatorname{ind}(ST) = \infty = \operatorname{ind} S + \operatorname{ind} T.$$

Let T be Fredhom, S be lower semi-Fredholm but not Fredholm. Then $\alpha(S) = \infty$ and since $\beta(T) < \infty$ then $\dim(T(X) \cap \ker S) = \infty$. Moreover, T maps $\ker(ST)$ onto $T(X) \cap \ker S$, thus $\alpha(ST) = \infty$. Therefore $\operatorname{ind} ST = \infty = \operatorname{ind} S + \operatorname{ind} T$, so the statement is proved if S, T are lower semi-Fredholm. If T and S are upper semi-Fredholm then the statements follows by duality. ∎

If $T \in \Phi_+(X, Y)$ the range $T(X)$ need not be complemented, and analogously if $T \in \Phi_-(X, Y)$ the kernel may be not complemented. The study of the following two classes of operators was initiated by Atkinson [**28**].

Definition 1.48. *If X and Y are Banach spaces then $T \in L(X, Y)$ is said to be* left Atkinson *if $T \in \Phi_+(X, Y)$ and $T(X)$ is complemented in X. The operator $T \in L(X, Y)$ is said to be* right Atkinson *if $T \in \Phi_-(X, Y)$ and* $\ker(T)$ *is complemented in Y. The class of left Atkinson operators and right Atkinson operators will be denoted by $\Phi_{\mathrm{l}}(X, Y)$ and $\Phi_{\mathrm{r}}(X, Y)$, respectively.*

Clearly,

$$\Phi(X, Y) \subseteq \Phi_{\mathrm{l}}(X, Y) \subseteq \Phi_+(X, Y),$$

and

$$\Phi(X, Y) \subseteq \Phi_{\mathrm{r}}(X, Y) \subseteq \Phi_-(X, Y).$$

Moreover,

$$\Phi(X, Y) = \Phi_{\mathrm{l}}(X, Y) \cap \Phi_{\mathrm{r}}(X, Y).$$

In order to give a precise description of Atkinson operators we need first to establish an useful lemma.

Lemma 1.49. *If $T \in \Phi_+(X,Y)$ and M is a closed subspace of X then $T(M)$ is closed.*

Proof Since ker T is finite-dimensional then ker T is complemented. Write $X = \ker T \oplus M_1$. Clearly, the restriction $T|M_1$ is injective and has closed range, since $T(X) = T(M_1)$. For some finite-dimensional subspace M_0 then we have $M = (M \cap M_1) + M_0$, and hence

$$T(M) = T(M \cap M_1) + T(M_0),$$

where $T(M \cap M_1)$ is closed, since $T|M_1$ is injective with closed range, and $T(M_0)$ is finite-dimensional. Hence $T(M)$ is closed. ∎

Theorem 1.50. *Let X, Y, and Z be Banach spaces and $T \in L(X,Y)$. Then the following assertions are equivalent:*

(i) $T \in \Phi_l(X,Y)$;

(ii) *there exists* $S \in L(Y,X)$ *such that* $I_X - ST \in \mathcal{F}(X)$;

(iii) *there exists* $S \in L(Y,X)$ *such that* $I_X - ST \in \mathcal{K}(X)$.

Analogously, the following assertions are equivalent:

(iv) $T \in \Phi_r(X,Y)$;

(v) *there exists* $S \in L(Y,X)$ *such that* $I_Y - TS \in \mathcal{F}(Y)$;

(vi) *there exists* $S \in L(Y,X)$ *such that* $I_Y - TS \in \mathcal{K}(Y)$.

Proof (i) ⇒ (ii) Suppose that $T \in \Phi_l(X,Y)$. Let $Q \in L(Y)$ be a projection of Y onto $T(X)$. Since ker T is finite-dimensional then ker T is complemented, so we can write $X = \ker T \oplus M$. Then the restriction $T|M : M \to T(X)$ is bijective. Let $S_0 : T(X) \to M$ denote the inverse of $T|M$ and set $S := S_0Q$. Clearly, $(I_x - ST)|M = 0$, thus $I_X - ST$ is a finite-dimensional operator.

(ii) ⇒ (iii) Obvious.

(iii) ⇒ (i) Suppose that there exists $S \in L(Y,X)$ such that $I_X - ST = K \in \mathcal{K}(X)$. Then ST is a Fredholm operator and hence, by Theorem 1.46, T is upper semi-Fredholm and S is lower semi-Fredholm. It remains to prove that $T(X)$ is complemented in Y. We have $I_X - K = ST$, thus by Corollary 1.44 $ST \in \Phi(X)$ and ST ha finite descent $q := q(ST)$. Set $M := (ST)^q(X)$ and observe that by Theorem 1.42 $(ST)^q \in \Phi(X)$, and hence M is a closed finite-codimensional subspace of X. Let P denote a projection of X onto M and write $S_0 := PS : Y \to M$. Then $S_0T|M = (ST)|M = I_M - K|M$, K_M compact and $(S_0T|M)(M) = M$. The last equality yields that $S_0T|M \in L(M)$ is onto and since $S_0T|M = I_M -$

$K|M$ from Corollary 1.44 we can deduce that $S_0T|M$ is also injective. Then $(S_0T|M)^{-1}S_0T|M = I_M$ and hence by Theorem 1.11 $T(M) = T|M(M)$ is complemented in Y. Since $T(X) = T(M) \oplus T(I-P)(X)$ and $T(I-P)(X)$ is finite-dimensional it then follows that $T(X)$ is a closed finite-codimensional subspace of Y, and hence complemented in Y.

The proof of the equivalences (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) follows in a similar way. ∎

Theorem 1.51. *If $T \in L(X,Y)$ then the following statements are equivalent:*

(i) *$T \in \Phi(X,Y)$;*

(ii) *there exists $S \in L(Y,X)$ such that $I_X - ST \in \mathcal{F}(X)$ and $I_Y - TS \in \mathcal{F}(Y)$;*

(iii) *there exists $S \in L(Y,X)$ such that $I_X - ST \in \mathcal{K}(X)$ and $I_Y - TS \in \mathcal{K}(Y)$.*

Proof This follows from Theorem 1.50, once observed that if $T \in \Phi(X,Y)$ then T is relatively regular, so $T(X)$ and $\ker T$ are complemented. ∎

Corollary 1.52. *Suppose that $T \in L(X)$. Then*

(i) *$T \in \Phi_{\rm r}(X)$ if and only if the class rest $\hat{T} = T + \mathcal{K}(X)$ is right invertible in the Calkin algebra $L(X)/\mathcal{K}(X)$.*

(ii) *$T \in \Phi_{\rm l}(X)$ if and only if the class rest $\hat{T} = T + \mathcal{K}(X)$ is left invertible in $L(X)/\mathcal{K}(X)$.*

(iii) *$T \in \Phi(X)$ if and only if the class rest $\hat{T} = T + \mathcal{K}(X)$ is invertible in $L(X)/\mathcal{K}(X)$.*

The characterization (iii) of Fredholm operators is known as the *Atkinson characterizaton* of Fredholm operators .

Remark 1.53. It should be noted that if X is an infinite-dimensional complex Banach space then $\lambda I - T \notin \Phi(X)$ for some $\lambda \in \mathbb{C}$. This follows from the classical result that the spectrum of an arbitrary element of a complex infinite-dimensional Banach algebra is always non-empty, see next Theorem 2.2. In fact, by the Atkinson characterization of Fredholm operators, $\lambda I - T \notin \Phi(X)$ if and only if $\widehat{T} := T + \mathcal{K}(X)$ is non-invertible in the Calkin algebra $L(X)/\mathcal{K}(X)$. An analogous result holds for semi-Fredholm operators on infinite-dimensional Banach spaces: if X is an infinite-dimensional Banach space and $T \in L(X)$ then $\lambda I - T \notin \Phi_+(X)$ (respectively, $\lambda I - T \notin \Phi_-(X)$) for some $\lambda \in \mathbb{C}$.

In the following theorem we list some other properties of Atkinson operators, the proof is left to the reader, see Problems IV.13 of Lay and Taylor [**75**].

Theorem 1.54. *If X, Y and Z are Banach spaces we have:*

(i) *If $T \in \Phi_{\rm l}(X, Y)$ and $S \in \Phi_{\rm l}(Y, Z)$ then $ST \in \Phi_{\rm l}(X, Z)$. Analogously, if $T \in \Phi_{\rm r}(X, Y)$ and $S \in \Phi_{\rm r}(Y, Z)$ then $ST \in \Phi_{\rm r}(X, Z)$. The sets $\Phi_{\rm l}(X)$, $\Phi_{\rm r}(X)$ and $\Phi(X)$ are semi-groups in $L(X)$.*

(ii) *Suppose that $T \in L(X, Y)$, $S \in L(Y, Z)$ and $ST \in \Phi_{\rm l}(X, Z)$. Then $S \in \Phi_{\rm l}(Y, Z)$. Analogously, suppose that $T \in L(X, Y)$, $S \in L(Y, Z)$ and $ST \in \Phi_{\rm r}(X, Z)$. Then $T \in \Phi_{\rm r}(X, Y)$.*

6. Some perturbation properties of semi-Fredholm operators

In order to give some perturbation results on the classes $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ we need a preliminary Lemma.

Lemma 1.55. *If $T \in L(X, Y)$ then $T \in \Phi_+(X, Y)$ if and only if there exists a closed finite-codimensional subspace M such that $T|M$ is bounded below.*

Proof If $T \in \Phi_+(X, Y)$ then the finite-dimensional kernel of T is complemented. Write $X = \ker T \oplus M$. Then the restriction $T|M$ is bijective, so it is bounded below.

Conversely, suppose that there exists a closed finite-codimensional subspace M such that $T|M$ is bounded below. Since $\ker T \cap M = \{0\}$ we deduce that $\ker T$ is finite-dimensional. Let N be a complement of M, i.e. $X = M \oplus N$ with $\dim N < \infty$. Then $T(X) = T(M) + T(N)$ with $T(M)$ closed by assumption. Since $T(N)$ is finite-dimensional, it then follows that $T(X)$ is closed. Hence $T \in \Phi_+(X, Y)$. ∎

The next theorem shows that the classes $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi(X, Y)$ are stable under compact perturbations.

Theorem 1.56. *If X and Y are Banach spaces the following statements hold:*

(i) *If $\Phi_+(X, Y) \neq \varnothing$ then $\Phi_+(X, Y) + \mathcal{K}(X, Y) \subseteq \Phi_+(X, Y)$.*

(ii) *If $\Phi_-(X, Y) \neq \varnothing$ then $\Phi_-(X, Y) + \mathcal{K}(X, Y) \subseteq \Phi_-(X, Y)$.*

(iii) *If* $\Phi(X,Y) \neq \varnothing$ *then* $\Phi(X,Y) + \mathcal{K}(X,Y) \subseteq \Phi(X,Y)$.

Proof Let $T \in \Phi_+(X,Y)$ and $K \in \mathcal{K}(X,Y)$. By Lemma 1.55 there exists a closed finite-codimensional subspace M_1 such that $T|M_1$ is bounded below, or equivalently by (4) the injectivity modulus $j(T|M_1) > 0$. Since K is compact by Theorem 1.39 there exists another closed finite-codimensional subspace M_2 such that $\|K|M_2\| < \frac{1}{2}j(T|M_1)$. Set $M := M_1 \cap M_2$. Clearly, M is finite-codimensional and closed. Moreover, from the estimate

$$\inf_{x\in M,\|x\|=1} \|(T+K)x\| \geq \inf_{x\in M,\|x\|=1} (\|Tx\| - \|Kx\|) \geq \frac{1}{2}j(T|M_1),$$

we see that $j(T+K) > 0$, thus the restriction $(T+K)|M$ is bounded below on M. By Lemma 1.55 then $T + K \in \Phi_+(X,Y)$, as desired.

(ii) This follows by duality from part (i), recalling that if $T \in \Phi_-(X,Y)$ then $T^* \in \Phi_+(Y^*,X^*)$ and K^* is still compact by Schauder theorem.

(iii) This follows from part (i) and part (ii). ∎

For every operator $T \in L(X,Y)$ define by $\overline{\beta}(T)$ the codimension of the closure of $T(X)$. Clearly $\overline{\beta}(T) \leq \beta(T)$, and if $\beta(T)$ is finite then $\overline{\beta}(T) = \beta(T)$, since $T(X)$ is closed by Corollary 1.8.

Theorem 1.57. *Let* $T \in L(X,Y)$ *be a bounded operator on a Banach space* X, Y*. Then the following assertions hold:*

(i) $T \in \Phi_+(X,Y)$ *if and only if* $\alpha(T-K) < \infty$ *for all compact operators* $K \in \mathcal{K}(X,Y)$.

(ii) $T \in \Phi_-(X,Y)$ *if and only if* $\overline{\beta}(T-K) < \infty$ *for all compact operators* $K \in \mathcal{K}(X,Y)$.

Proof (i) Since $\Phi_+(X,Y)$ is stable under compact perturbations, we have only to show that if $\alpha(T-K) < \infty$ for all compact operators $K \in \mathcal{K}(X,Y)$ then $T \in \Phi_+(X,Y)$.

Suppose that $T \notin \Phi_+(X,Y)$. Then T does not have a bounded inverse, so there exists $x_1 \in X$ with $\|x_1\| = 1$, such that $\|Tx_1\| \leq \frac{1}{2}$. By the Hahn Banach theorem we can find an element $f_1 \in X^*$ such that $\|f_1\| = 1$ and $f_1(x_1) = 1$. Let us consider a bi-orthogonal system $\{x_k\}$ and $\{f_k\}$ such that

$$\|x_k\| = 1, \quad \|Tx_k\| \leq 2^{1-2k}, \quad \text{and} \quad \|f_k\| \leq 2^{k-1} \quad \text{for all } k = 1, 2, \dots n-1.$$

Since the restriction of T to the closed subspace $N := \bigcap_{k=1}^{n-1} \ker f_k$ does not admit a bounded inverse, there is an element $x_n \in N$ such that $\|x_n\| = 1$ and $\|Tx_n\| \leq 2^{1-2n}$. Let $g \in X^*$ be such that $g(x_n) = 1$ and $\|g\| = 1$. Define $f_n \in X^*$ by the assignment:

$$f_n := g - \sum_{k=1}^{n-1} g(x_k) f_k.$$

Then $f_n(x_k) = \delta_{nk}$ for $k = 1, 2, \dots, n$ and $\|f_n\| \leq 2^{n-1}$. By means of an inductive argument we can construct two sequences $(x_k) \subset X$ and $(f_k) \subset X^*$ such that

$$\|x_k\| = 1, \quad \|f_k\| \leq 2^{k-1}, \quad f_k(x_j) = \delta_{kj}, \quad \text{and} \quad \|Tx_k\| \leq 2^{1-2k}.$$

Define $K_n \in L(X, Y)$ by

$$K_n(x) := \sum_{k=1}^{n} f_k(x) T x_k \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly K_n is a finite-dimensional operator for every n, and for $n > m$ the following estimate holds:

$$\|(K_n - K_m)x\| \leq \sum_{k=m+1}^{n} 2^{k-1} 2^{1-2k} \|x\|,$$

so $\|K_n - K_m\| \to 0$ as $m, n \to \infty$. If we define

$$Kx := \sum_{k=1}^{\infty} f_k(x) T x_k$$

then $K_n \to K$ in the operator norm, so K is compact. Moreover, $Kx = Tx$ for any $x = x_k$ and this is also true for any linear combination of the elements x_k. Since these elements are linearly independent we then conclude that $\alpha(T - K) = \infty$. Hence the equivalence (i) is proved.

(ii) Also here one direction of the equivalence is immediate: suppose that $T \in \Phi_-(X, Y)$ and $K \in \mathcal{K}(X, Y)$. By duality we then have $T^* \in \Phi_+(Y^*, X^*)$ and $K^* \in \mathcal{K}(Y^*, X^*)$. From this we obtain that $T^* - K^* \in \Phi_+(Y^*, X^*)$, and therefore, again by duality, $T - K \in \Phi_-(X, Y)$.

Suppose that $T \notin \Phi_-(X, Y)$. Then either $T(X)$ is closed and $\overline{\beta}(T) = \infty$ or $T(X)$ is not closed. In the first case taking $K = 0$ we obtain $\overline{\beta}(T - K) = \infty$ and we are finished. So assume that $T(X)$ is not closed.

Let (a_n) be a sequence of integers defined inductively by

$$a_1 := 2, \quad a_n := 2\left(1+\sum_{k=1}^{n-1} a_k\right), \quad \text{for } n = 2, 3, \dots$$

We prove now that there exists a sequence (y_k) in Y and a sequence (f_k) in Y^* such that, for all $k \in \mathbb{N}$, we have

$$\text{(8)} \quad \|y_k\| \leq a_k, \quad \|f_k\| = 1, \quad \|T^*(f_k)\| < \frac{1}{2^k a_k}, \quad \text{and} \quad f_j(y_k) = \delta_{jk}.$$

We proceed by induction. Since $T(X)$ is not closed, also $T^*(Y^*)$ is not closed, by Theorem 1.2. Hence there exists $f_1 \in Y^*$ such that $\|f_1\| = 1$ and $\|T^* f_1\| < \frac{1}{4}$, and there exists $y_1 \in Y$ such that $\|y_1\| < 2$ and $f_1(y_1) = 1$.

Suppose that there exist $y_1, y_2, \dots, y_{n-1}$ in Y and $f_1, f_2, \dots f_{n-1}$ in Y^* such that (8) hold for every $j = 2, \dots, n-1$. Let $f_n \in Y^*$ be such

$$f_n(y_k) = 0 \quad \text{for } k = 1, \dots, n-1, \quad \|f_n\| = 1, \quad \|T^* f_n\| < \frac{1}{2^n a_n}.$$

There exists also $y \in Y$ such that $f_n(y) = 1$ and $\|y\| < 2$. Define

$$y_n := y - \sum_{k=1}^{n-1} f_k(y) y_k.$$

Then

$$\|y_n\| \leq \|y\| \left(1 + \sum_{k=1}^{n-1} \|y_k\|\right) \leq 2\left(1+\sum_{k=1}^{n-1} a_k\right) = a_n,$$

by the induction hypothesis. Furthermore, from the choice of the elements y_n and f_n we see that $f_n(y_n) = 1$ while $f_n(y_k) = 0$ for all $k = 1, 2, \dots, n$. Finally, $f_k(y_n) = f_k(y) - f_k(y) = 0$ for all $k = 1, 2, \dots, n$, so y_n has the desired properties.

By induction then there exist the two sequences sequence (y_k) in Y and a sequence (f_k) in Y^* which satisfy (8). We now define

$$K_n x := \sum_{k=1}^{n} (T^* f_k)(x) y_k \quad \text{for } n \in \mathbb{N}.$$

For $n > m$ we obtain

$$\|K_n x - K_m x\| \leq \sum_{m+1}^{n} \|T^* f_k\| \|x\| \|y_k\| \leq \left(\sum_{m+1}^{n} \frac{1}{2^k}\right) \|x\| \leq \frac{\|x\|}{2^m},$$

and this implies that the finite-dimensional operator K_n converges to the compact operator $K \in \mathcal{K}(X, Y)$ defined by

$$Kx := \sum_{k=1}^{\infty} (T^* f_k)(x) y_k.$$

For each $x \in X$ and each k we have $f_k(Kx) = T^* f_k(x) = f_k(Tx)$, so each of the f_k annihilates $(T - K)(X)$. Since the f_k are linearly independent we then conclude that $\overline{\beta}(T - K) = \infty$. ∎

Theorem 1.58. *Let $T \in \Phi_{\pm}(X, Y)$. Then ind$(T + K) =$ ind$T$ for all $K \in \mathcal{K}(X, Y)$.*

Proof By Theorem 1.56 the set $\Phi_{\pm}(X, Y)$ is stable under compact perturbations. Suppose first $T \in \Phi(X, Y)$. According part (iii) of Theorem 1.51 there exists $S \in L(Y, X)$ and $K_1 \in \mathcal{K}(X)$ such that $ST = I_X - K_1$, and hence, by Corollary 1.37 and the index theorem,

$$0 = \text{ind}\,(I_X - K_1) = \text{ind}\, ST = \text{ind}\, T + \text{ind}\, S,$$

from which we obtain $\text{ind}\, T = -\text{ind}\, S$. Moreover, since

$$S(T + K) = I_X + (-K_1 + SK)$$

and $K_1 + SK$ is compact it then follows that $\text{ind}\, S + \text{ind}(T + K) = 0$. Therefore,

$$\text{ind}(T + K) = -\text{ind}\, S = \text{ind}\, T.$$

If T is upper semi-Fredholm but not Fredholm then also $T + K$ is upper semi-Fredholm but not Fredholm, so $\text{ind}(T + K) = \text{ind}\, T = -\infty$. For the other case, that T is lower semi-Fredholm but not Fredholm, proceed in a similar way. ∎

Lemma 1.59. *Suppose that for $T \in L(X)$, X a Banach space, we have $\|T\| < 1$. Then $I - T$ is invertible.*

Proof We have

$$\| \sum_{j=m}^{n} T^j \| \leq \sum_{j=m}^{n} \|T^j\| \leq \sum_{j=m}^{n} \|T\|^j,$$

so $\sum_{n=0}^{\infty} T^n$ is a Cauchy series in the Banach algebra $L(X)$, and hence there is an $S \in L(X)$ such that $S = \sum_{n=0}^{\infty} T^n$. From this it then follows that

$$ST = TS = \sum_{n=0}^{\infty} T^{n+1} = S - I,$$

from which we obtain $(I-T)S = S-TS = I$ and $S(I-T) = S-ST = I$. Therefore, $I - T$ is invertible with inverse $S = \sum_{n=0}^{\infty} T^n$. ■

We show now that $\Phi(X, Y)$ is stable under *small* perturbations.

Theorem 1.60. *For every $T \in \Phi(X, Y)$, X and Y Banach spaces, there exists $\rho := \rho(T) > 0$ such that for all $S \in L(X, Y)$ with $\|S\| < \rho$ then $T + S \in \Phi(X, Y)$ and ind$(T + S) =$ ind T. The set $\Phi(X, Y)$ is open in $L(X, Y)$.*

Proof By Theorem 1.51 there exists $U \in L(Y, X)$ and $K \in \mathcal{K}(X)$ such that $UT = I_X - K$. By Corollary 1.44 we now that $I_X - K = UT$ is a Fredholm operator having index 0. By Theorem 1.46 and by the index theorem we then deduce that U is a Fredholm operator and $\operatorname{ind} T = -\operatorname{ind} U$. Now, take $S \in L(X, Y)$ such that

$$\|S\| < \rho := \frac{1}{\|U\|}.$$

Then $\|US\| \leq \|U\|\|S\| < 1$, thus, by Lemma 1.59, $I_X + US$ is invertible, in particular a Fredholm operator having index 0. By Theorem 1.58 it then follows that

$$\operatorname{ind}(I_X + US - K) = 0.$$

We have

$$U(T + S) = UT + US = (I_X - K) + US = (I_X + US) - K,$$

so $\operatorname{ind} U(T + S) = 0$, and since U is a Fredholm operator then $T + S \in \Phi(X, Y)$, always by Theorem 1.46, and

$$\operatorname{ind} U + \operatorname{ind}(T + S) = 0.$$

Therefore, $\operatorname{ind}(T + S) = -\operatorname{ind} U = \operatorname{ind} T$. ■

We want show now that also the classes of semi-Fredholm operators are stable under small perturbations. We need first to give some information on the gap between closed linear subspaces of a Banach space. Let M and N be two closed linear subspaces of a Banach space X and define, if $M \neq \{0\}$,

$$\delta(M, N) := \sup\{\operatorname{dist}(x, N) :, x \in M, \|x\| = 1\},$$

while $\delta(M, N) = 0$ if $M = \{0\}$. The *gap* between M and N is defined as

$$\Theta(M, N) := \max\{\delta(M, N), \delta(N, M)\}.$$

It is clear that $0 \leq \Theta(M,N) \leq 1$, while $\Theta(M,N) = 0$ precisely when $M = N$ and $\Theta(M,N) = \Theta(N,M)$. Moreover, $\Theta(M,N) = \Theta(M^\perp, N^\perp)$.

Theorem 1.61. *If $\Theta(M,N) < 1$, then either M and N are both infinite-dimensional or $\dim M = \dim N < \infty$.*

The proof of Theorem 1.61 requires the well-know Borsuk-Ulam theorem, see [**40**]. For further information on gap theory see Kato [**71**].

Theorem 1.62. *Suppose that $T \in L(X,Y)$. Then we have*

(i) *If $T \in \Phi_+(X,Y)$ then there exists $\varepsilon > 0$ such that for every $S \in L(X,Y)$ for which $\|S\| < \varepsilon$ we have $T + S \in \Phi_+(X,Y)$. Moreover, $\alpha(T+S) \leq \alpha(T)$ and $\operatorname{ind}(T+S) = \operatorname{ind} T$.*

(ii) *If $T \in \Phi_-(X,Y)$ then there exists $\varepsilon > 0$ such that for every $S \in L(X,Y)$ for which $\|S\| < \varepsilon$ we have $T + S \in \Phi_-(X,Y)$. Moreover, $\beta(T+S) \leq \beta(T)$ and $\operatorname{ind}(T+S) = \operatorname{ind} T$.*

Proof (i) Suppose $T \in \Phi_+(X,Y)$. Then $\ker T$ admits a topological complement, since it has finite dimension. Write $X = \ker T \oplus M$. Then $T|M$ is bounded below with closed range $T(X) = T(M)$, so there exists $K > 0$ such that $\|Tx\| \geq K\|x\|$ for all $x \in M$. Let $\varepsilon := \frac{K}{3}$. If $\|S\| < \varepsilon$ then

$$0 \leq \|Sx\| < \frac{K}{3}\|x\|,$$

and hence

$$\|(T+S)x\| \geq \|Tx\| - \|Sx\| \geq \frac{2K}{3}\|x\| \quad \text{for all } x \in M. \tag{9}$$

This shows that the restriction $(T+S)|M$ is bounded below, i.e. $(T+S)|M$ is injective and $(T+S)(M)$ is closed. Set $n := \alpha(T)$ and suppose that $\alpha(T+S) > n$. Then $\ker(T+S)$ has at least $n+1$ linearly independent vectors, say $x_1, x_2, \dots x_{n+1}$. Since

$$\ker(T+S) \cap M = \ker((T+S)|M) = \{0\}$$

then the corresponding elements $\hat{x}_1, \hat{x}_2, \dots \hat{x}_{n+1}$ in the quotient $\hat{X} := X/M$ are linearly independent. But from the decomposition $X = \ker T \oplus M$ we see that $\dim X/M = \dim \ker T = n$, a contradiction. Therefore, $\alpha(T+S) \leq \alpha(T)$.

We want show now that $(T+S)(X)$ is a closed subspace. Evidently,

$X = \ker T + [M \oplus \ker(T+S)]$, so one can find a finite dimensional subspace N such that $X = N \oplus [M \oplus \ker(T+S)]$, from which we obtain

$$(T+S)(X) = (T+S)(N) + (T+S)(M \oplus \ker(T+S)).$$

Since, $(T+S)(N)$ is finite-dimensional, in order to prove that $(T+S)(X)$ is closed it suffices to prove that $(T+S)(M \oplus \ker(T+S))$ is closed. But

$$(T+S)(M \oplus \ker(T+S)) = (T+S)(M)$$

and in this equality the last subspace is closed, as observed above. Thus $T+S \in \Phi_+(X,Y)$.

To show the index equality $\operatorname{ind}(T+S) = \operatorname{ind} T$, suppose first $\beta(T) < \infty$. Then $T \in \Phi(X,Y)$, so by Theorem 1.60 the equality $\operatorname{ind}(T+S) = \operatorname{ind} T$ holds. Suppose the other case, i.e. $\beta(T) = \infty$. If $x \in M$ from the estimate

$$\|Tx - (T+S)x\| = \|Sx\| \leq \varepsilon\|x\| \leq \frac{K}{3}\|x\| \leq \frac{1}{3}\|Tx\|,$$

and using (9) we get for all $x \in M$

$$\|Tx - (T+S)x\| = \|Sx\| \ \leq \|S\|\|x\| \leq \frac{3\|S\|}{2K}\|(T+S)x\| < \frac{1}{2}.$$

The above inequalities shows that the gap between $Y_1 := (T+S)(M)$ and $Y_2 := T(M) = T(X)$ is less than $\frac{1}{2}$, so

$$\Theta({Y_1}^\perp, {Y_2}^\perp) = \Theta(Y_1, Y_2) < \frac{1}{2},$$

hence by Theorem 1.61 we have $\dim {Y_1}^\perp = \dim {Y_2}^\perp = \beta(T)$, since $Y_2 = T(X)$. Finally, since $(T+S)(X) = Y_1 \oplus W$, for some finite-dimensional subspace W, we then conclude that $\beta(T+S) = \beta(T) = \infty$. Hence the statement is proved for upper semi-Fredholm operators.

(ii) The statement for lower semi-Fredholm operators follows by duality. ∎

Corollary 1.63. *The sets $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi(X,Y)$ are open subsets of $L(X,Y)$. The index is constant on every connected components of $\Phi_\pm(X,Y)$.*

Proof The first statement is clear by Theorem 1.62. Let Γ be an open maximal connected component of the open set $\Phi_\pm(X,Y)$. Fix an operator $T_0 \in \Gamma$. By Theorem 1.62 then the index function $T \to \operatorname{ind} T$ is continuous on Γ. Therefore the set $\Gamma_1 := \{T \in \Gamma : \operatorname{ind} T = \operatorname{ind} T_0\}$ is both open and closed. This implies that $\Gamma = \Gamma_1$. ∎

If the perturbation of $T \in \Phi_\pm(X)$ is caused by a multiple of the identity we have the so-called *punctured neighborhood theorem*:

Theorem 1.64. *Let $T \in \Phi_+(X)$. Then there exists $\varepsilon > 0$ such that*

$$\alpha(\lambda I + T) \leq \alpha(T) \quad \text{for all } |\lambda| < \varepsilon,$$

and $\alpha(\lambda I - T)$ is constant for all $0 < |\lambda| < \varepsilon$.
Analogously, if $T \in \Phi_-(X)$ then there exists $\varepsilon > 0$ such that

$$\beta(\lambda I + T) \leq \beta(T) \quad \text{for all } |\lambda| < \varepsilon,$$

and $\beta(\lambda I + T)$ is constant for all $0 < |\lambda| < \varepsilon$.

Proof For the constancy of the functions $\alpha(\lambda I - T)$ and $\beta(\lambda I - T)$ on the punctured disc $0 < |\lambda| < \varepsilon$ see Kato [**71**, Theorem 5.31]. ■

7. Further and recent developments

In this section we shall introduce some classes of operators which are related to the classes $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$. We do not give any proof of the results established, we just refer for a proof to the original sources or to the books [**1**] and [**84**]. Hence this section is essentially a review on recent extensions of semi-Fredholm theory.

Semi-Fredholm operators on Banach spaces admit an important decomposition property introduced by Kato in [**69**]. To see this let first introduce the following class of operators intensively studied by Mbekhta, [**78**], [**79**], by Mbekhta and Ouahab [**82**] .

Definition 1.65. *A bounded operator $T \in L(X)$, X a Banach space, is said to be a* semi-regular *if T has closed range $T(X)$ and* ker $T \subseteq$ *$T^n(X)$ for every $n \in \mathbf{N}$.*

By Theorem 1.24 if T is semi-regular then $C(T) = T^\infty(X)$. Clearly, bounded below as well as surjective operators are semi-regular. Note that the condition ker $T \subseteq T^n(X)$ is equivalent to the conditions listed in Theorem 1.15 and Corollary 1.16. For a proof of the following result see [**1**, Therem 1.31].

Theorem 1.66. *Let $T \in L(X)$ be semi-regular. Then $\lambda I - T$ is semi-regular for all $|\lambda| < \gamma(T)$, where $\gamma(T)$ denotes the reduced minimal modulus.*

The product of two semi-regular operators, also commuting semi-regular operators, need not be semi-regular, see Example 1.27 in [**1**]. However, we have (see [**1**, Theorem 1.26]):

Theorem 1.67. *If TS is semi-regular and $TS = ST$ then both T and S are semi-regular.*

The punctured neighborhood theorem for semi-Fredholm operators motivates the following definition.

Definition 1.68. *Let $T \in \Phi_\pm(X)$, X a Banach space. Let $\varepsilon > 0$ as in Theorem 1.64. If $T \in \Phi_+(X)$ the* jump $j(T)$ *is defined by*

$$j(T) := \alpha(T) - \alpha(\lambda I + T), \quad 0 < |\lambda| < \varepsilon,$$

while, if $T \in \Phi_-(X)$, the jump $j(T)$ is defined by

$$j(T) := \beta(T) - \beta(\lambda I + T), \quad 0 < |\lambda| < \varepsilon.$$

The continuity of the index ensures that both definitions of the jump coincide whenever T is a Fredholm operator.

In general, semi-Fredholm are not semi-regular. In fact we have (see for a proof [**1**, Theorem 1.58])

Theorem 1.69. *A semi-Fredholm operator $T \in L(X)$ is semi-regular precisely when $j(T) = 0$.*

Definition 1.70. *An operator $T \in L(X)$, X a Banach space, is said to admit a* generalized Kato decomposition, *abbreviated as GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent.*

Clearly, every semi-regular operator has a GKD $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD $M = \{0\}$ and $N = X$.

A relevant case is obtained if we assume in the definition above that $T|N$ is nilpotent, i.e. there exists $d \in \mathbb{N}$ for which $(T|N)^d = 0$. In this case T is said to be of *Kato type of operator of order d*.

An operator is said to be *essentially semi-regular* if it admits a GKD (M, N) such that N is finite-dimensional. Note that if T is essentially semi-regular then $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent.

Hence we have the following implications:

$$\begin{aligned} T \text{ semi-regular} \quad &\Rightarrow \quad T \text{ essentially semi-regular } \Rightarrow T \text{ of Kato type} \\ &\Rightarrow \quad T \;\text{ admits a GKD.} \end{aligned}$$

The proof of the following theorem may be found in [**1**, Theorem 1.41 and Theorem 1.42].

Theorem 1.71. *Suppose that (M, N) is a GKD for $T \in L(X)$. Then $K(T) = K(T|M)$ and $K(T)$ is closed. If T is of Kato type then $K(T) = T^{\infty}(X)$.*

The following result was first observed by Kato [**69**] , see for a proof Aiena [**1**, Chapter 1] or Müller [**84**, Chapter III].

Theorem 1.72. *Every semi-Fredholm operator $T \in L(X)$ is essentially semi-regular.*

If $T \in L(X)$ is of Kato type then $\lambda I - T$ is semi-regular on a punctured disc of 0 ([**1**, Theorem 1.44]):

Theorem 1.73. *If $T \in L(X)$ is of Kato type then there exists $\varepsilon > 0$ such that $\lambda I - T$ is semi-regular for all $0 < |\lambda| < \varepsilon$.*

The complemented version of semi-regular operators is given by Saphar operators, see Chapter II of [**84**] and Schmoeger [**99**].

Definition 1.74. *A bounded operator $T \in L(X)$ is said to be* Saphar *if T is both semi-regular and relatively regular.*

A perturbation type theorem holds also for Saphar operators ([**84**, Lemma 6, Chapter II]:

Theorem 1.75. *If $T \in L(X)$ is Saphar then there exists $\varepsilon > 0$ such that $T + S$ is relatively regular for every $S \in L(X)$ such that $\|S\| < \varepsilon$ and $ST = TS$.*

The following concept was first introduced by Grabiner [**64**].

Definition 1.76. *Let $T \in L(X)$ and $d \in \mathbb{N}$. T is said to have* uniform descent for $n \geq d$ *if $T(X) + \ker T^n = T(X) + \ker T^d$ for all $n \geq d$. If, in addition, $T(X) + \ker T^d$ is closed then T is said to have* topological uniform descent for $n \geq d$.

Note that if either of the quantities $\alpha(T)$, $\beta(T)$, $p(T)$, $q(T)$ is finite then T has uniform descent. Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1.77. *$T \in L(X)$ is said to be* quasi-Fredholm of degree d, *if there exists $d \in \mathbb{N}$ such that:*

(a) *$dis(T) = d$,*

(b) $T^n(X)$ *is a closed subspace of* X *for each* $n \geq d$,
(c) $T(X) + \ker T^d$ *is a closed subspace of* X.

It is easily seen that every Fredholm operator is quasi-Fredholm. Let $QF(d)$ denote the class of all quasi-Fredholm of degree d. It is easily seen that if $T \in QF(d)$ then $T^* \in QF(d)$. The following characterization of quasi-Fredholm operators is due to Berkani [**31**].

Theorem 1.78. *If* X *is a Banach space then* $T \in QF(d)$ *if and only if there exists* $n \in \mathbb{N}$ *such that* $T^n(X)$ *is closed and the restriction* $T|T^n(X)$ *is semi-regular.*

Observe that quasi-Fredholm operators are precisely all operators $T \in L(X)$ having topological uniform descent $n \geq d$ and such that $T^{d+1}(X)$ is closed, see for details [**83**]. An example of operator that is not quasi-Fredholm but has topological uniform descent may be found in [**31**].

A natural question is whenever a quasi-Fredholm operator is of Kato type. The following result has been recently proved by Müller [**85**]. In the special case of Hilbert spaces operators these results were proved in [**30**], [**32**].

Theorem 1.79. *Let* $T \in L(X)$, X *a Banach space, be quasi-Fredholm of degree* d. *Suppose that* $T(X) + \ker T^d$ *and* $\ker T \cap T^d(X)$ *are complemented. Then* T *is of Kato type.*

Note that in a Hilbert space every closed subspace is complemented, so by Theorem 1.79 we have

Theorem 1.80. *Every quasi-Fredholm operator acting on a Hilbert space is of Kato type.*

The concept of semi-Fredholm operator may be generalized as follows. For every $T \in L(X)$ and a nonnegative integer n, let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). The following class of operators has been introduced by Berkani *et al.* ([**30**], [**36**] and [**32**]).

Definition 1.81. $T \in L(X)$ *is said to be* semi B-Fredholm , *(resp.,* B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer* $n \geq 0$ *the range* $T^n(X)$ *is closed and* $T_{[n]}$ *is a Fredholm operator (resp., Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case* $T_{[m]}$ *is a semi-Fredholm operator for all* $m \geq n$ *(*[**36**]*). This enables one to define the index of a semi B-Fredholm as* $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

By Proposition 2.5 of [**36**] every semi B-Fredholm operator on a Banach space is quasi-Fredholm. Note that T is B-Fredholm if and only if T^* is B-Fredholm. In this case $\operatorname{ind} T^* = -\operatorname{ind} T$.

Theorem 1.82. *If $T \in L(X)$, X a Banach space, then the following statements hold:*

(i) *T is B-Fredholm if and only if there exist two closed invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is Fredholm and $T|N$ is nilpotent.*

(ii) *T is B-Fredholm of index 0 if and only if there exist two closed invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is Fredholm having index 0 and $T|N$ is nilpotent.*

The following punctured disc theorem is a particular case of a result proved in [**36**, Corollary 3.2] for operators having topological uniform descent for $n \geq d$.

Theorem 1.83. *Suppose that $T \in L(X)$ is upper semi B-Fredholm. Then there exists an open disc $\mathbb{D}(0, \varepsilon)$ centered at 0 such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ and*

$$ind\,(\lambda I - T) = ind\,(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Moreover, if $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ then

$$\alpha(\lambda I - T) = dim\,(\ker T \cap T^d(X)) \quad \textit{for some } d \in \mathbb{N},$$

so that $\alpha(\lambda I - T)$ is constant as λ ranges in $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$\alpha(\lambda I - T) \leq \alpha(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Analogously, if $T \in L(X)$ is lower semi B-Fredholm then there exists an open disc $\mathbb{D}(0, \varepsilon)$ centered at 0 such that $\lambda I - T$ is lower semi-Fredholm for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$ind\,(\lambda I - T) = ind\,(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Moreover, if $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ then

$$\beta(\lambda I - T) = codim\,(\ker T^d + T^(X)) \quad \textit{for some } d \in \mathbb{N},$$

so that $\beta(\lambda I - T)$ is constant as λ ranges in $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$\beta(\lambda I - T) \leq \beta(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

The product of commuting semi B-Fredholm operators T, $S \in L(X)$ and $TS = ST$ then TS is also semi B-Fredholm [**32**, Theorem 3.2], with $\operatorname{ind} TS = \operatorname{ind} T + \operatorname{ind} S$. In particular, if T is semi B-Fredholm then T^n is semi B-Fredholm for all $n \in \mathbb{N}$. As a particular case of a perturbation result proved by Grabiner [**64**] for operators having topological uniform descent we have:

Theorem 1.84. *If T, $S \in L(X)$, X a Banach space, then the following statements hold:*

(i) *If T and S are B-Fredholm, $TS = ST$ and $\|T - S\|$ is sufficiently small then $ind T = ind S$.*

(ii) *If T is B-Fredholm, $ST = TS$, $\|T - S\|$ is sufficiently small and $T - S$ is invertible, then $S \in \Phi(X)$ with $ind T = ind S$.*

B-Fredholm operator on Banach spaces are stable under finite rank perturbations (not necessarily commuting) (see [**31**, Prop. 2.7] and [**32**, Theorem 3.2]. Precisely we have :

Theorem 1.85. *If $T \in L(X)$ is B-Fredholm then $T + K$ is also B-Fredholm for all $K \in \mathcal{F}(X)$ and $ind(T + K) = ind T$.*

The class of B-Fredholm operator is not stable under compact perturbations. For instance, if $K \in \mathcal{K}(X)$ is such that $K^n(X)$ is not closed for all $n \in \mathbb{N}$ then K is not a B-Fredholm operator. Let F be finite-dimensional operator. Clearly, F is B-Fredholm, but $K + F$ is not B-Fredholm, otherwise $K = K + F - F$ would be B-Fredholm.

CHAPTER 2

The single-valued extension property

This chapter deals with an important property for bounded linear operators acting on complex infinite-dimensional Banach spaces, the so-called *single-valued extension property* (SVEP). This property appeared first in Dunford [**48**], and has received a systematic treatment in the classical Dunford and Schwartz book [**50**]. After the pioneering work of Dunford this property has revealed to be a basic tool in local spectral theory, see the books Colojoară, C. Foiaş [**42**] , Vasilescu [**102**] , and the more recent books by Laursen, Neumann [**76**] and by Aiena [**1**].

The SVEP is satisfied by several important classes of operators, as normal operators in Hilbert spaces, by compact operators and by convolution operators on group algebras. Mainly, we concern with a localized version of the SVEP, the SVEP at a point introduced by Finch [**54**] and more recently studied by several authors, Mbektha ([**80**] [**81**]), Aiena *et al* ([**14**], [**12**], [**15**]). In particular, we relate the localized SVEP to the finiteness of ascent and of the descent. Hence this chapter may be viewed as the part of this book in which the interaction between local spectral theory and Fredholm theory comes into focus. In fact, for semi-Fredholm operators the SVEP at a point may be characterized in several ways. Some of these characterizations involve two important spectral subspaces, the quasi-nilpotent part and the analytic core of an operator. In this chapter we introduce other important classes of semi-Fredholm operators: Weyl operators and Browder operators. The spectral properties of compact operators are then used to define the class of Riesz operators.

1. Isolated points of the spectrum

We begin by establishing some preliminary notions on the functional calculus of bounded operators on Banach spaces. We first begin with basic notions on the spectrum of an operator.

Definition 2.1. *Let $T \in L(X)$, X a complex Banach space. The* spectrum *of T is defined as the set*

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}.$$

Recall that if $T \in L(X)$ is bijective then the inverse T^{-1} is still bounded. If $T \in L(X)$ then the *spectral radius* of T is defined by

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

The proof of the following standard result may be found in any text of functional analysis, see for instance [**68**, Theorem 45.1]

Theorem 2.2. *For every $T \in L(X)$, X a complex Banach space, we have*

$$\sigma(T) \neq \varnothing \quad \text{and} \quad r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Moreover, $\sigma(T)$ is a compact subset of $\mathbb{C}$.

Note that $\sigma(T) = \sigma(T^*)$. Set $\rho(T) := \mathbb{C}\setminus\sigma(T)$. The set $\rho(T)$ is called the *resolvent set* of T, while the map $R(\lambda, T) : \lambda \in \rho(T) \to (\lambda I - T)^{-1}$ is called the *resolvent* of T.

Theorem 2.3. *If $T \in L(X)$ and $S \in L(X)$ commute then*

$$r(T + S) \leq r(T) + r(S) \quad \text{and } r(TS) \leq r(T)r(S).$$

Proof See [**68**, Proposition 45.1]. ∎

Remark 2.4. The results of Theorem 2.2 and Theorem 2.3 are valid in a more general setting: If x is an element of a complex *Banach algebra* $\mathcal{A}$ with unit u then the *spectrum* of x, defined as

$$\sigma(x) := \{\lambda \in \mathbb{C} : \lambda u - x \text{ is not invertible}\}$$

is a not empty compact subset of $\mathbb{C}$. Furthermore, if x, y are two commuting elements of $\mathcal{A}$ then the spectral radius satisfies the inequalities

$$r(x + y) \leq r(x) + r(y) \quad \text{and} \quad r(xy) \leq r(x)r(y),$$

see Proposition 45.1 of ([**68**].

In this chapter we shall also consider two distinguished parts of the spectrum, the *approximate point spectrum*, defined as

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

and the *surjectivity spectrum*, defined as

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

The approximate point spectrum is a non-empty set:

Theorem 2.5. *If $T \in L(X)$ then $\sigma_{\rm a}(T)$ and $\sigma_{\rm s}(T)$ are non-empty compact subsets of $\mathbb{C}$. Indeed, both spectra contain the boundary $\partial\sigma(T)$ of $\sigma(T)$. Furthermore,*

$$\sigma_{\rm a}(T) = \sigma_{\rm s}(T^*) \quad \text{and} \quad \sigma_{\rm s}(T) = \sigma_{\rm a}(T^*). \tag{10}$$

Proof The equalities (10) are consequences of part (i) Theorem 1.6. The proof of the inclusion $\partial\sigma(T) \subseteq \sigma_{\rm a}(T)$ may be found in [**1**, Theorem 2.42]. Since $\partial\sigma(T) = \partial\sigma(T^*)$ we then have $\partial\sigma(T) \subseteq \sigma_{\rm a}(T^*) = \sigma_{\rm s}(T)$. Clearly, again by Theorem 1.6, part (ii), $\sigma_{\rm a}(T)$ and $\sigma_{\rm s}(T)$ are compact subsets of $\mathbb{C}$. ∎

According the classical concepts of complex analysis, a vector-valued function $f : \Delta \to X$, X a complex Banach space and Δ a non-empty open subset of the complex field $\mathbb{C}$, is said to be *locally analytic* if it is differentiable at every point of Δ, i.e. if $\lambda_0 \in \Delta$ then there exists an $f'(\lambda_0) \in X$ such that

$$\|\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)\| \to 0 \quad \text{as } \lambda \to \lambda_0.$$

If Δ is connected and f is locally analytic then f is said to be *analytic*. ¿From Theorem 44.1 of [**68**] if $\lambda_0, \lambda \in \rho(T)$ then

$$\|\frac{|R(\lambda, T) - R(\lambda_0, T)}{\lambda - \lambda_0}\| \leq \frac{\|R(\lambda_0, T)\|^2}{1 - |\lambda - \lambda_0|\|R(\lambda_0, T)\|}$$

so the function $R(\lambda, T)$ depends continuously from λ, and the resolvent function is analytic on $\rho(T)$. Many of the theorems, together their proofs, of complex function theory may be transfered in a purely formal way to vector-valued locally analytic functions (as Cauchy's theorems and integral formula, Taylor and Laurent expansions, Liouville's theorem). For instance, if Γ is an *integration path*, i.e. an oriented, closed, rectifiable curve in $\mathbb{C}$, for every continuous function $f : \Gamma \to X$ the *path integral* is defined, as in the classical case, as the limit of Riemann sums.

$$\int_\Gamma f(\lambda)\, d\lambda := \lim \sum f(\xi_k)(\lambda_k - \lambda_{k-1}).$$

The chain lengths of $(\lambda I - T)$ are intimately related to the poles of the resolvent $R(\lambda, T)$. If $f : \mathbb{D}(\lambda_0, \delta) \setminus \{\lambda_0\} \to X$, X a Banach space, is a analytic function defined in the open disc centered at λ_0 with values in X, then, by the *Laurent expansion*, f can be represented in the form

$$f(\lambda) = \sum_{k=0}^{\infty} a_k(\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(\lambda - \lambda_0)^k},$$

with $a_k, b_k \in X$, and $\lambda \in \mathbb{D}(\lambda_0, \delta) \setminus \{\lambda_0\}$. The coefficients are given by the formulas

$$a_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{(\lambda - \lambda_0)^{k+1}} d\lambda, \quad b_k = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - \lambda_0)^{k-1} d\lambda,$$

where Γ is a positively oriented circle $|\lambda - \lambda_0| = r$, with $0 < r < \delta$, see Proposition 46.7 of [**68**] for details. We say that λ_0 is a *pole of order* p if $b_p \neq 0$ and $b_n = 0$ for all $n > p$.

Let λ_0 be an isolated point of $\sigma(T)$ and let us consider the particular case of the Laurent expansion of the analytic function $R_\lambda : \lambda \in \rho(T) \to (\lambda I - T)^{-1} \in L(X)$ in a neighborhood of λ_0. According the previous considerations, we have

$$R_\lambda = \sum_{k=0}^{\infty} Q_k(\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} \frac{P_k}{(\lambda - \lambda_0)^k} \quad \text{with } P_k, Q_k \in L(X).$$

for all $0 < |\lambda - \lambda_0| < \delta$. The coefficients are calculated according the formulas

$$Q_k = \frac{1}{2\pi i} \int_\Gamma \frac{R_\lambda}{(\lambda - \lambda_0)^{k+1}} d\lambda \tag{11}$$

$$P_k = \frac{1}{2\pi i} \int_\Gamma R_\lambda (\lambda - \lambda_0)^{k-1} d\lambda, \tag{12}$$

where Γ is a sufficiently small, positively oriented circle around λ_0.

Let $\mathcal{H}(\sigma(T))$ be the set of all complex-valued functions which are locally analytic on an open set containing $\sigma(T)$. Suppose that $f \in \mathcal{H}(\sigma(T))$, $\Delta(f)$ be the domain of f, and let Γ denote a contour in $\Delta(f)$ that surrounds $\sigma(T)$. This means a positively oriented finite system $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of closed rectifiable curves in $\Delta(f) \setminus \sigma(T)$ such that $\sigma(T)$ is contained in the inside of Γ and $\mathbb{C} \setminus \Delta(f)$ in the outside of Γ. Then

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

is well-defined and does not depend on the particular choice of Γ. It should be noted that *mutatis mutandis* all the arguments and notions introduced above may be extended to Banach algebras with unit u: if

$a \in \mathcal{H}(\sigma(a))$ is an analytic function defined on open set containing $\sigma(a)$ then

$$f(a) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda u - a)^{-1} d\lambda,$$

is defined in a similar way as we have done for the elements of the Banach algebra $L(X)$.

In the particular case that of functions which are equal to 1 on certain parts of $\sigma(T)$ and equal to 0 on others we get idempotent operators. To see this, suppose that σ is a spectral set (i.e. σ and $\sigma(T) \setminus \sigma$ are both closed) and $\Delta := \Delta_1 \cup \Delta_2$ is an open covering of $\sigma(T)$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ and $\sigma \subseteq \Delta_1$, define $h(\lambda) := 1$ for λ on Δ_1 and $h(\lambda) := 0$ for λ on Δ_2. Consider the operator $P_\sigma := h(T)$. It is easy to check that $P_\sigma^2 = P_\sigma$, so P_σ is a projection called the *spectral projection associated with* σ, and obviously

$$P_\sigma = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} d\lambda, \tag{13}$$

where Γ is a curve enclosing σ which separate σ from the remaining part of the spectrum.

Let us consider again the case of an isolated point λ_0 of $\sigma(T)$. Then $\{\lambda_0\}$ is a spectral set so we can consider the spectral projection associated with $\{\lambda_0\}$. It is easy to check that if P_k are defined according (12) then

$$P_1 = P_0, \quad P_k = (T - \lambda_0 I)^{n-1} P_0 \quad (k = 1, 2, \dots) \tag{14}$$

The equation (14) show that either $P_k \neq 0$, or that there exists a natural p such that $P_k \neq 0$ for $k = 1, \dots, p$ but $P_k = 0$ for $k > p$. In the second case the isolated point λ_0 is pole of order p of T.

The spectral sets produce the following decomposition see [**68**, §49].

Theorem 2.6. *If σ is a spectral set (possibly empty) of $T \in L(X)$ then the projection in (13) generates the decomposition $X = P_\sigma(X) \oplus \ker P_\sigma$. The subspaces $P_\sigma(X)$ and $\ker P_\sigma$ are invariant under every $f(T)$ with $f \in \mathcal{H}(\sigma(T))$; the spectrum of the restriction $T|P_\sigma(X)$ is σ and the spectrum of $T|\ker P_\sigma$ is $\sigma(T) \setminus \sigma$.*

The proof of the following basic result may be found in [**68**, Proposition 50.2].

Theorem 2.7. *Let $T \in L(X)$. Then $\lambda_0 \in \sigma(T)$ is a pole of $R(\lambda, T)$ if and only if $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$. Moreover, if $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ then p is the order of the pole, every pole*

$\lambda_0 \in \sigma(T)$ is an eigenvalue of T, and if P_0 is the spectral projection associated with $\{\lambda_0\}$ then

$$P_0(X) = \ker\,(\lambda_0 I - T)^p, \quad \ker\, P_0 = (\lambda_0 I - T)^p(X).$$

In the case that $\lambda_0 I - T$ is a Fredholm operator having both ascent and descent finite we have much more, see Heuser [**68**, Proposition 50.3].

Theorem 2.8. *If $\lambda_0 \in \sigma(T)$ then $\lambda_0 I - T$ is a Fredholm operator having both ascent and descent finite if and only if λ_0 is an isolated spectral point of T and the corresponding spectral projection P_0 is finite-dimensional.*

In the following result, due to Schmoeger [**98**] we show that for an isolated point λ_0 of $\sigma(T)$ the quasi-nilpotent part $H_0(\lambda_0 I - T)$ and the analytical core $K(\lambda_0 I - T)$ may be precisely described as a range or a kernel of a projection.

Theorem 2.9. *Let $T \in L(X)$, where X is a Banach space, and suppose that λ_0 is an isolated point of $\sigma(T)$. If P_0 is the spectral projection associated with $\{\lambda_0\}$, then:*

(i) *$P_0(X) = H_0(\lambda_0 I - T)$;*

(ii) *$\ker\, P_0 = K(\lambda_0 I - T)$.*

In particular, if $\{\lambda_0\}$ is a pole of the resolvent, or equivalently $p := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$, then

$$P_0(X) = H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$\ker P_0 = K(\lambda_0 I - T) = (\lambda_0 I - T)^p(X).$$

Proof (i) Since λ_0 is an isolated point of $\sigma(T)$ there exists a positively oriented circle $\Gamma := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \delta\}$ which separates λ_0 from the remaining part of the spectrum. We have

$$(\lambda_0 I - T)^n P_0 x = \frac{1}{2\pi i} \int_\Gamma (\lambda_0 - \lambda)^n (\lambda I - T)^{-1} x \, d\lambda \quad \text{for all } n = 0, 1, \cdots .$$

Now, assume that $x \in P_0(X)$. We have $P_0 x = x$ and it is easy to verify the following estimate:

$$\|(\lambda_0 I - T)^n x\| \le \frac{1}{2\pi} 2\pi \delta^{n+1} \max_{\lambda \in \Gamma} \|(\lambda I - T)^{-1}\| \|x\|.$$

Obviously this estimate also holds for some $\delta_o < \delta$, and consequently

$$\limsup \|(\lambda_0 I - T)^n x\|^{1/n} < \delta. \tag{15}$$

This proves the inclusion $P_0(X) \subseteq H_0(\lambda_0 I - T)$.

Conversely, assume that $x \in H_0(\lambda_0 I - T)$ and hence that the inequality (15) holds. Let $S \in L(X)$ denote the operator

$$S := \frac{1}{\lambda_0 - \lambda}(\lambda_0 I - T).$$

Evidently the Neumann series

$$\sum_{n=0}^{\infty} S^n x = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_0 - \lambda}(\lambda_0 I - T) \right)^n x$$

converges for all $\lambda \in \Gamma$. If y_λ denotes its sum for every $\lambda \in \Gamma$, from a standard argument of functional analysis we obtain that $(I - S)y_\lambda = x$. A simple calculation also shows that

$$y_\lambda = (\lambda - \lambda_0)(\lambda I - T)^{-1} x$$

and therefore

$$(\lambda I - T)^{-1} x = -\sum_{n=0}^{\infty} \frac{(\lambda_0 I - T)^n x}{(\lambda_0 - \lambda)^{n+1}} \quad \text{for all } \lambda \in \Gamma.$$

A term by term integration then yields

$$P_0 x = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} x \, d\lambda = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{(\lambda_0 - \lambda)} x \, d\lambda = x,$$

and this proves the inclusion $H_0(\lambda_0 I - T) \subseteq P_0(X)$. This completes the proof of the equality (i).

(ii) There is no harm in assuming that $\lambda_0 = 0$. We have $\sigma(T|P_0(X)) = \{0\}$, and $0 \in \rho(T|\ker P_0)$. ¿From the equality $T(\ker P_0) = \ker P_0$ we obtain $\ker P_0 \subseteq K(T)$, see Theorem 1.29.

It remains to prove the reverse inclusion $K(T) \subseteq \ker P_0$. To see this we first show that $H_0(T) \cap K(T) = \{0\}$. This is clear because $H_0(T) \cap K(T) = K(T|H_0(T))$, and the last subspace is $\{0\}$ since the restriction of T on the Banach space $H_0(T)$ is a quasi-nilpotent operator, see Corollary 2.32. Hence $H_0(T) \cap K(T) = \{0\}$. From this it then follows that

$$\begin{aligned} K(T) &= K(T) \cap X = K(T) \cap [\ker P_0 \oplus P_0(X)] \\ &\subseteq \ker P_0 + K(T) \cap H_0(T) = \ker P_0, \end{aligned}$$

so the desired inclusion is proved.

The last assertion is clear from Theorem 2.7. ∎

2. SVEP

The basic importance of the single-valued extension property arises in connection with some basic notions of local spectral theory. Before introducing the typical tools of this theory, and in order to give a first motivation, let us begin with some considerations on spectral theory.

As already observed the resolvent function $R(\lambda, T) := (\lambda I - T)^{-1}$ of $T \in L(X)$, X a Banach space, is an analytic operator-valued function defined on the resolvent set $\rho(T)$. Setting

$$f_x(\lambda) := R(\lambda, T)x \quad \text{for any } x \in X,$$

the vector-valued analytic function $f_x : \rho(T) \to X$ satisfies the equation

$$(\lambda I - T) f_x(\lambda) = x \quad \text{for all } \lambda \in \rho(T). \tag{16}$$

It is possible to find analytic solutions of the equation $(\lambda I - T) f_x(\lambda) = x$ for some (sometimes even for all) values of λ that are in the spectrum of T. To see this, let $T \in L(X)$ be a bounded operator on a Banach space X such that the spectrum $\sigma(T)$ has a non-empty spectral subset $\sigma \neq \sigma(T)$. If $P_\sigma := P(\sigma, T)$ is the spectral projection of T associated with σ then, by Theorem 2.6, $\sigma(T \mid P_\sigma(X)) = \sigma$, so the restriction $(\lambda I - T) \mid P_\sigma(X)$ is invertible for all $\lambda \notin \sigma$.

Let $x \in P_\sigma(X)$. Then the equation (16) has the analytic solution

$$g_x(\lambda) := (\lambda I - T \mid P_\sigma(X))^{-1} x \quad \text{for all } \lambda \in \mathbb{C} \setminus \sigma.$$

This property suggests the following concepts:

Definition 2.10. *Given an arbitrary operator $T \in L(X)$, X a Banach space, let $\rho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $\mathcal{U}_\lambda$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U}_\lambda \to X$ such that the equation*

$$(\mu I - T) f(\mu) = x \quad \textit{holds for all } \mu \in \mathcal{U}_\lambda. \tag{17}$$

If the function f is defined on the set $\rho_T(x)$ then it is called a local resolvent function *of T at x. The set $\rho_T(x)$ is called the* local resolvent *of T at x. The* local spectrum *$\sigma_T(x)$ of T at the point $x \in X$ is defined to be the set*

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

Evidently $\rho_T(x)$ is the open subset of $\mathbb{C}$ given by the union of the domains of all the local resolvent functions. Moreover,

$$\rho(T) \subseteq \rho_T(x) \quad \text{and} \quad \sigma_T(x) \subseteq \sigma(T).$$

It is immediate to check the following elementary properties of $\sigma_T(x)$:

(a) $\sigma_T(0) = \varnothing$;

(b) $\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y)$ for all $x, y \in X$;

(c) $\sigma_{(\lambda I - T)}(x) \subseteq \{0\}$ if and only if $\sigma_T(x) \subseteq \{\lambda\}$.

Furthermore,

(d) $\sigma_T(Sx) \subseteq \sigma_T(x)$ for every $S \in L(X)$ which commutes with T. In fact, let $f : \mathcal{U}_\lambda \to X$ be an analytic function on the open set $\mathcal{U}_\lambda \subseteq \mathbb{C}$ for which $(\mu I - T)f(\mu) = x$ holds for all $\mu \in \mathcal{U}_\lambda$. If $TS = ST$ then the function $S \circ f : \mathcal{U}_\lambda \to X$ is analytic and satisfies the equation

$$(\mu I - T)S \circ f(\mu) = S((\mu I - T)f(\mu)) = Sx \quad \text{for all } \mu \in \mathcal{U}_\lambda,$$

Therefore $\rho_T(x) \subseteq \rho_T(Sx)$ and hence $\sigma_T(Sx) \subseteq \sigma_T(x)$.

Definition 2.11. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single-valued extension property *at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighborhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

In the sequel we collect some basic properties of the SVEP.

(a) The SVEP ensures the consistency of the local solutions of equation (17), in the sense that if $x \in X$ and T has the SVEP at $\lambda_0 \in \rho_T(x)$ then there exists a neighborhood $\mathcal{U}$ of λ_0 and an *unique* analytic function $f : \mathcal{U} \to X$ satisfying the equation $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$.

Another important consequence of the SVEP is the existence of a maximal analytic extension $\tilde{f}$ of $R(\lambda, T)x := (\lambda I - T)^{-1}x$ to the set $\rho_T(x)$ for every $x \in X$. This function identically verifies the equation

$$(\mu I - T)\tilde{f}(\mu) = x \quad \text{for every } \mu \in \rho_T(x)$$

and, obviously,

$$\tilde{f}(\mu) = (\mu I - T)^{-1}x \quad \text{for every } \mu \in \rho(T).$$

(b) It is immediate to verify that the SVEP is inherited by the restrictions on closed invariant subspaces, i.e., if $T \in L(X)$ has the

SVEP at λ_0 and M is a closed T-invariant subspace, then $T|M$ has the SVEP at λ_0. Moreover,

$$\sigma_T(x) \subseteq \sigma_{T|M}(x) \quad \text{for every } x \in M.$$

(c) Let $\sigma_{\rm p}(T)$ denote the *point spectrum* of $T \in L(X)$, i.e.,

$$\sigma_{\rm p}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

It is easy to see from definition of localized SVEP the following implication:

$$\sigma_{\rm p}(T) \;\text{ does not cluster at } \lambda_0 \Rightarrow \; T \;\text{ has the SVEP } \; \lambda_0.$$

Indeed, if $\sigma_p(T)$ does not cluster at λ_0 then there is an neighborhood $\mathcal{U}$ of λ_0 such that $\lambda I - T$ is injective for every $\lambda \in \mathcal{U}$, $\lambda \neq \lambda_0$.

Let $f : \mathcal{V} \to X$ be an analytic function defined on another neighborhood $\mathcal{V}$ of λ_0 for which the equation $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathcal{V}$. Obviously we may assume that $\mathcal{V} \subseteq \mathcal{U}$. Then $f(\lambda) \in \ker\,(\lambda I - T) = \{0\}$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$, and hence $f(\lambda) = 0$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$. From the continuity of f at λ_0 we conclude that $f(\lambda_0) = 0$. Hence $f \equiv 0$ in $\mathcal{V}$ and therefore T has the SVEP at λ_0. The same argument shows that

$$\sigma_{\rm a}(T) \;\text{ does not cluster at } \lambda_0 \Rightarrow \; T \;\text{ has the SVEP } \; \lambda_0,$$

and hence by duality,

$$\sigma_{\rm s}(T) \;\text{ does not cluster at } \lambda_0 \Rightarrow \; T^* \;\text{ has the SVEP } \; \lambda_0,$$

(d) From part (c) every operator T has the SVEP at an isolated point of the spectrum. Obviously T has the SVEP at every $\lambda \in \rho(T)$. From these facts it follows that every quasi-nilpotent operator T has the SVEP. More generally, if $\sigma_{\rm p}(T)$ has empty interior then T has the SVEP. In particular, any operator with a real spectrum has the SVEP. Note that there are examples of operators T having SVEP and such that $\sigma_{\rm p}(T) \neq \varnothing$.

Definition 2.12. *For every subset* Ω *of* $\mathbb{C}$ *the* local spectral subspace *of* T *associated with* Ω *is the set*

$$X_T(\Omega) := \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

Obviously, if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$.
In the next theorem we collect some of the basic properties of the subspaces $X_T(\Omega)$, see [**1**, Chapter 2].

Theorem 2.13. *Let $T \in L(X)$, X a Banach space, and Ω every subset of $\mathbb{C}$. Then the following properties hold:*

(i) $X_T(\Omega)$ *is a linear T-hyper-invariant subspace of X, i.e., for every bounded operator S that commutes with T we have $S(X_T(\Omega)) \subseteq X_T(\Omega)$;*

(ii) $X_T(\Omega) = X_T(\Omega \cap \sigma(T))$*;*

(iii) *If $\lambda \notin \Omega$, $\Omega \subseteq \mathbb{C}$, then $(\lambda I - T)(X_T(\Omega)) = X_T(\Omega)$;*

(iv) *Suppose that $\lambda \in \Omega$ and $(\lambda I - T)x \in X_T(\Omega)$ for some $x \in X$. Then $x \in X_T(\Omega)$;*

(v) *For every family $(\Omega_j)_{j \in J}$ of subsets of $\mathbb{C}$ we have*

$$X_T(\bigcap_{j \in J} \Omega_j) = \bigcap_{j \in J} X_T(\Omega_j);$$

(vi) *If Y is a T-invariant closed subspace of X for which $\sigma(T\,|Y) \subseteq \Omega$, then $Y \subseteq X_T(\Omega)$. In particular, $Y \subseteq X_T(\sigma(T\,|Y))$ holds for every closed T-invariant closed subspace of X.*

We have already observed that the null operator 0 has an empty local spectrum. The next theorem (for a proof see [**1**, Theorem 2.8]) shows that if T has the SVEP then 0 is the *unique* element of X having empty local spectrum. Actaually, this property characterizes the SVEP.

Theorem 2.14. *Let $T \in L(X)$, X a Banach space. Then the following statements are equivalent :*

(i) *T has the SVEP;*

(ii) $X_T(\varnothing) = \{0\}$*;*

(iii) $X_T(\varnothing)$ *is closed.*

A bounded operator $Q \in L(X)$ is said to be *quasi-nilpotent* if $\lambda I - T$ is invertible for all $\lambda \neq 0$, i.e. $\sigma(T) = \{0\}$. The SVEP is inherited under commuting quasi-nilpotent operators, see [**1**, Corollary 2.12]:

Corollary 2.15. *Suppose that $T \in L(X)$ has the SVEP and Q is quasi-nilpotent commuting with T. Then $T + Q$ has the SVEP.*

We next show that the SVEP is preserved by some transforms.

Definition 2.16. *An operator $U \in L(X, Y)$ between the Banach spaces X and Y is said to be a* quasi-affinity *if U is injective and has dense range. The operator $S \in L(Y)$ is said to be a* quasi-affine transform *of $T \in L(X)$ if there is a quasi-affinity $U \in L(Y, X)$ such that $TU = US$.*

Theorem 2.17. *If $T \in L(X)$ has the SVEP at $\lambda_0 \in \mathbb{C}$ and $S \in L(Y)$ is a quasi-affine transform of T then S has the SVEP at λ_0.*

Proof Let $f : \mathcal{U} \to Y$ be an analytic function defined on an open neighborhood $\mathcal{U}$ of λ_0 such that $(\mu I - S)f(\mu) = 0$ for all $\mu \in \mathcal{U}$. Then $U(\mu I - S)f(\mu) = (\mu I - T)Uf(\mu) = 0$ and the SVEP of T at λ_0 entails that $Uf(\mu) = 0$ for all $\mu \in \mathcal{U}$. Since U is injective then $f(\mu) = 0$ for all $\mu \in \mathcal{U}$, hence S has the SVEP at λ_0. ■

We now give a simple characterization of the elements of the analytic core $K(T)$ by means of the local resolvent $\rho_T(x)$.

Theorem 2.18. *Let $T \in L(X)$, X a Banach space. Then*

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \in \rho_T(x)\}.$$

Proof Let $x \in K(T)$. We can suppose that $x \neq 0$. According to the definition of $K(T)$, let $\delta > 0$ and $(u_n) \subset X$ be a sequence for which

$$x = u_0, \; Tu_{n+1} = u_n, \quad \|u_n\| \le \delta^n \|x\| \quad \text{for every } n \in \mathbb{Z}_+.$$

Then the function $f : \mathbb{D}(0, 1/\delta) \to X$, where $\mathbb{D}(0, 1/\delta)$ is the open disc centered at 0 and radius $1/\delta$, defined by

$$f(\lambda) := -\sum_{n=1}^{\infty} \lambda^{n-1} u_n \quad \text{for all } \lambda \in \mathbb{D}(0, 1/\delta),$$

is analytic and verifies the equation $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathbb{D}(0, 1/\delta)$. Consequently $0 \in \rho_T(x)$.

Conversely, if $0 \in \rho_T(x)$ then there exists an open disc $\mathbb{D}(0, \varepsilon)$ and an analytic function $f : \mathbb{D}(0, \varepsilon) \to X$ such that

$$(\lambda I - T)f(\lambda) = x \quad \text{for every } \lambda \in \mathbb{D}(0, \varepsilon). \tag{18}$$

Since f is analytic on $\mathbb{D}(0, \varepsilon)$ there exists a sequence $(u_n) \subset X$ such that

$$f(\lambda) = -\sum_{n=1}^{\infty} \lambda^{n-1} u_n \quad \text{for every } \lambda \in \mathbb{D}(0, \varepsilon). \tag{19}$$

Clearly $f(0) = -u_1$. and taking $\lambda = 0$ in (18) we obtain

$$Tu_1 = -T(f(0)) = x.$$

On the other hand

$$x = (\lambda I - T)f(\lambda) = Tu_1 + \lambda(Tu_2 - u_1) + \lambda^2(Tu_3 - u_2) + \cdots$$

for all $\lambda \in \mathbb{D}(0, \varepsilon)$. Since $x = Tu_1$ we conclude that

$$Tu_{n+1} = u_n \quad \text{for all } n = 1, 2, \cdots .$$

Hence letting $u_0 = x$ the sequence (u_n) satisfies for all $n \in \mathbb{Z}_+$ the first of the conditions which define $K(T)$.

It remains to prove the condition $\|u_n\| \leq \delta^n \|x\|$ for a suitable $\delta > 0$ and for all $n \in \mathbb{Z}_+$. Take $\mu > 1/\varepsilon$. Since the series (19) converges then $|\lambda|^{n-1}\|u_n\| \to 0$ as $n \to \infty$ for all $\|\lambda\| < \varepsilon$ and, in particular, $(1/\mu^{n-1})\|u_n\| \to 0$, so that there exists a $c > 0$ such that

$$\|u_n\| \leq c\, \mu^{n-1} \quad \text{for every } n \in \mathbb{N}. \tag{20}$$

¿From the estimates (20) we easily obtain

$$\|u_n\| \leq \left(\mu + \frac{c}{\|x\|}\right)^n \|x\|$$

and therefore $x \in K(T)$. ■

3. Quasi-nilpotent part of an operator

Another important invariant subspace for a bounded operator $T \in L(X)$, X a Banach space, is defined as follows :

Definition 2.19. *Let $T \in L(X)$, X a Banach space. The* quasi-nilpotent part *of T is defined to be the set*

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Clearly $H_0(T)$ is a linear subspace of X, generally not closed. In the following theorem we collect some elementary properties of $H_0(T)$.

Lemma 2.20. *For every $T \in L(X)$, X a Banach space, we have:*
(i) ker $(T^m) \subseteq \mathcal{N}^\infty(T) \subseteq H_0(T)$ *for every $m \in \mathbb{N}$;*
(ii) $x \in H_0(T) \Leftrightarrow Tx \in H_0(T)$*;*
(iii) ker $(\lambda I - T) \cap H_0(T) = \{0\}$ *for every $\lambda \neq 0$.*

Proof (i) If $T^m x = 0$ then $T^n x = 0$ for every $n \geq m$.

(ii) If $x_0 \in H_0(T)$ from the inequality $\|T^n Tx\| \leq \|T\|\|T^n x\|$ it easily follows that $Tx \in H_0(T)$. Conversely, if $Tx \in H_0(T)$ from

$$\|T^{n-1}Tx\|^{1/n-1} = (\|T^n x\|^{1/n})^{n/n-1}$$

we conclude that $x \in H_0(T)$.

(iii) If $x \neq 0$ is an element of $\ker\, (\lambda I - T)$ then $T^n x = \lambda^n x$, so

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = \lim_{n\to\infty} |\lambda| \|x\|^{1/n} = |\lambda|$$

and therefore $x \notin H_0(T)$. ∎

Theorem 2.21. *Let X be a Banach space. Then $T \in L(X)$ is quasi-nilpotent if and only if $H_0(T) = X$.*

Proof If T is quasi-nilpotent then $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, so that from $\|T^n x\| \leq \|T^n\| \|x\|$ we obtain that $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$ for every $x \in X$. Conversely, assume that $H_0(T) = X$. By the n-th root test the series

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{|\lambda|^{n+1}},$$

converges for each $x \in X$ and $\lambda \neq 0$. Define

$$y := \sum_{n=0}^{\infty} \frac{T^n x}{\lambda^{n+1}}.$$

It is easy to verify that $(\lambda I - T)y = x$, thus $(\lambda I - T)$ is surjective for all $\lambda \neq 0$. On the other hand, for every $\lambda \neq 0$ we have that

$$\{0\} = \ker\, (\lambda I - T) \cap H_0(T) = \ker\, (\lambda I - T) \cap X = \ker\, (\lambda I - T),$$

which shows that $\lambda I - T$ is invertible and therefore $\sigma(T) = \{0\}$. ∎

For the proof of the following result see in [**1**, Theorem 2.20].

Theorem 2.22. *If $T \in L(X)$ then*

$$H_0(T) \subseteq X_T(\{0\}) = \{x \in X : \sigma_T(x) \subseteq \{0\}\}. \tag{21}$$

If $T \in L(X)$ has the SVEP then $H_0(T) = X_T(\{0\})$.

The next theorem shows some elementary relationships between the analytic core and the quasi-nilpotent part of an operator.

Theorem 2.23. *For every bounded operator $T \in L(X)$, X a Banach space, we have:*

(i) $H_0(T) \subseteq^{\perp} K(T^*)$ *and* $K(T) \subseteq^{\perp} H_0(T^*)$*;*

(ii) *If T is semi-Fredholm , then*

$$\overline{H_0(T)} = \overline{\mathcal{N}^\infty(T)} =^{\perp} K(T^*) \quad \text{and } K(T) =^{\perp} H_0(T^*). \tag{22}$$

Proof (i) Consider an element $u \in H_0(T)$ and $f \in K(T^*)$. From the definition of $K(T^*)$ we know that there exists $\delta > 0$ and a sequence (g_n), $n \in \mathbb{Z}_+$ of X^* such that

$$g_0 = f, \quad T^* g_{n+1} = g_n \quad \text{and} \quad \|g_n\| \leq \delta^n \|f\|$$

for every $n \in \mathbb{Z}_+$. These equalities entail that $f = (T^*)^n g_n$ for every $n \in \mathbb{Z}_+$, so that

$$f(u) = (T^*)^n g_n(u) = g_n(T^n u) \quad \text{for every} \quad n \in \mathbb{Z}_+.$$

¿From that it follows that $|f(u)| \leq \|T^n u\| \|g_n\|$ for every $n \in \mathbb{Z}_+$ and therefore

$$|f(u)| \leq \delta^n \|f\| \|T^n u\| \quad \text{for every } n \in \mathbb{Z}_+. \tag{23}$$

¿From $u \in H_0(T)$ we now obtain that $\lim_{n\to\infty} \|T^n u\|^{1/n} = 0$ and hence by taking the n-th root in (23) we conclude that $f(u) = 0$. Therefore $H_0(T) \subseteq^\perp K(T^*)$.

The inclusion $K(T) \subseteq^\perp H_0(T^*)$ is proved in a similar way.

(ii) Assume that $T \in \Phi_\pm(X)$. Then T^* is semi-Fredholm, hence has closed range and $T^{*n}(X^*)$ is closed for all $n \in \mathbb{N}$. From the first part we also know that

$$\overline{\mathcal{N}^\infty(T)} \subseteq \overline{H_0(T)} \subseteq^\perp \overline{K(T^*)} =^\perp K(T^*),$$

since ${}^\perp K(T^*)$ is closed.

To show the first two equalities of (22) we need only to show the inclusion ${}^\perp K(T^*) \subseteq \overline{\mathcal{N}^\infty(T)}$. For every $T \in L(X)$ and every $n \in \mathbb{N}$ we have $\ker T^n \subseteq \mathcal{N}^\infty(T)$, and hence

$$\mathcal{N}^\infty(T)^\perp \subseteq \ker T^{n\perp} = T^{*n}(X^*)$$

because the last subspaces are closed for all $n \in \mathbb{N}$.

¿From this we easily obtain that $\overline{\mathcal{N}^\infty(T)}^\perp \subseteq T^{*\infty}(X^*) = K(T^*)$, where the last equality follows from Corollary 1.45. Consequently ${}^\perp K(T^*) \subseteq \overline{\mathcal{N}^\infty(T)}$, thus the equalities (22) are proved.

The equality $K(T) =^\perp H_0(T^*)$ is proved in a similar way. ∎

4. Localized SVEP

We have seen in Theorem 2.14 that T has the SVEP precisely when for every element $0 \neq x \in X$ we have $\sigma_T(x) \neq \varnothing$. The next fundamental result establishes a localized version of this result.

Theorem 2.24. *Suppose that $T \in L(X)$, X a Banach space. Then the following conditions are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) $\ker\,(\lambda_0 I - T) \cap X_T(\varnothing) = \{0\}$;

iii) $\ker\,(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;

(iv) *For every $0 \neq x \in \ker\,(\lambda_0 I - T)$ we have $\sigma_T(x) = \{\lambda_0\}$.*

Proof By replacing T with $\lambda_0 I - T$ we may assume without loss of generality $\lambda_0 = 0$.

(i) $\Leftrightarrow$ (ii) Assume that for $x \in \ker T$ we have $\sigma_T(x) = \varnothing$. Then $0 \in \rho_T(x)$, so there is an open disc $\mathbb{D}(0,\varepsilon)$ and an analytic function $f : \mathbb{D}(0,\varepsilon) \to X$ such that $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathbb{D}(0,\varepsilon)$. Then

$$T((\lambda I - T)f(\lambda)) = (\lambda I - T)T(f(\lambda)) = Tx = 0$$

for every $\lambda \in \mathbb{D}(0,\varepsilon)$. Since T has the SVEP at 0 then $Tf(\lambda) = 0$, and therefore $T(f(0)) = x = 0$.

Conversely, suppose that for every $0 \neq x \in \ker T$ we have $\sigma_T(x) \neq \varnothing$. Let $f : \mathbb{D}\ (0,\varepsilon) \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$ for every $\lambda \in \mathbb{D}\ (0,\varepsilon)$. Then $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$ for a suitable sequence $(u_n) \subset X$. Clearly $Tu_0 = T(f(0)) = 0$, so $u_0 \in \ker T$. Moreover, from the equalities $\sigma_T(f(\lambda)) = \sigma_T(0) = \varnothing$ for every $\lambda \in \mathbb{D}\ (0,\varepsilon)$ we obtain that

$$\sigma_T(f(0)) = \sigma_T(u_0) = \varnothing,$$

and therefore by the assumption we conclude that $u_0 = 0$. For all $0 \neq \lambda \in \mathbb{D}\ (0,\varepsilon)$ we have

$$0 = (\lambda I - T)f(\lambda) = (\lambda I - T)\sum_{n=1}^{\infty} \lambda^n u_n = \lambda(\lambda I - T)\sum_{n=1}^{\infty} \lambda^n u_{n+1},$$

and therefore

$$0 = (\lambda I - T)(\sum_{n=0}^{\infty} \lambda^n u_{n+1}) \quad \text{for every } 0 \neq \lambda \in \mathbb{D}\ (0,\varepsilon).$$

By continuity this is still true for every $\lambda \in \mathbb{D}\ (0,\varepsilon)$. At this point, by using the same argument as in the first part of the proof, it is possible to show that $u_1 = 0$, and by iterating this procedure we conclude that $u_2 = u_3 = \cdots = 0$. This shows that $f \equiv 0$ on $\mathbb{D}\ (0,\varepsilon)$, and therefore T has the SVEP at 0.

(ii) ⇔ (iii) It suffices to prove the equality

$$\ker T \cap K(T) = \ker T \cap X_T(\varnothing).$$

To see this observe first that by the proof of Theorem 2.22 we have $\ker T \subseteq H_0(T) \subseteq X_T(\{0\})$. From Theorem 2.18 it follows that

$$\ker T \cap K(T) = \ker T \cap X_T(\mathbb{C} \setminus \{0\}) \subseteq X_T(\{0\}) \cap X_T(\mathbb{C} \setminus \{0\}) = X_T(\varnothing).$$

Since $X_T(\varnothing) \subseteq X_T(\mathbb{C} \setminus \{0\}) = K(T)$ we then conclude that

$$\ker T \cap K(T) = \ker T \cap K(T) \cap X_T(\varnothing) = \ker T \cap X_T(\varnothing),$$

as required.

(ii) ⇒ (iv) Since $\ker T \subseteq H_0(T)$, from Theorem 2.22 it then follows that $\sigma_T(x) \subseteq \{0\}$ for every $0 \neq x \in \ker T$. By assumption $\sigma_T(x) \neq \varnothing$, so $\sigma_T(x) = \{0\}$.

(iv) ⇒ (ii) Obvious. ∎

Clearly, if $\lambda_0 I - T$ is injective then T has the SVEP at λ_0. The next result shows that if $\lambda_0 I - T$ is surjective and T has the SVEP at λ_0 then λ_0 belongs to the resolvent $\rho(T)$.

Corollary 2.25. *Let $T \in L(X)$, X a Banach space, be such that $\lambda_0 I - T$ is surjective. Then T has the SVEP at λ_0 if and only if $\lambda_0 I - T$ is injective.*

Proof We can assume $\lambda_0 = 0$. Assume that T is onto and has the SVEP at 0. Then $K(T) = X$ and by Theorem 2.24 $\ker T \cap X = \ker T = \{0\}$, so T is injective. The converse is clear. ∎

An immediate consequence of Corollary 2.25 is that every unilateral left shift on the Hilbert space $\ell_2(\mathbb{N})$ fails to have SVEP at 0. In the next chapter we shall see that other examples of operators which do not have SVEP are semi-Fredholm operators on a Banach space having index strictly greater than 0.

The following theorem shows that the surjectivity spectrum of an operator is closely related to the local spectra.

Theorem 2.26. *For every operator $T \in L(X)$ on a Banach space X we have*

$$\sigma_{\mathrm{s}}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Proof If $\lambda \notin \bigcup_{x\in X} \sigma_T(x)$ then $\lambda \in \rho_T(x)$ for every $x \in X$ and hence, directly from the definition of $\rho_T(x)$, we conclude that $(\lambda I - T)y = x$ always admits a solution for every $x \in X$, $\lambda I - T$ is surjective. Thus $\lambda \notin \sigma_{\rm s}(T)$.

Conversely, suppose $\lambda \notin \sigma_{\rm s}(T)$. Then $\lambda I - T$ is surjective and therefore $X = K(\lambda I - T)$. From Theorem 2.18 it follows that $0 \notin \sigma_{\lambda I-T}(x)$ for every $x \in X$, and consequently $\lambda \notin \sigma_T(x)$ for every $x \in X$. ■

Let us define

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \text{ has no SVEP at } \lambda\}.$$

Clearly, $\Xi(T) = \varnothing$ if T has SVEP and from the identity theorem of analytic functions it easily follows that $\Xi(T)$ is an open set, and consequently is contained in the interior of $\sigma(T)$.

Corollary 2.27. *If X is a Banach space and $T \in L(X)$ then $\sigma(T) = \Xi(T) \cup \sigma_{\rm s}(T)$. In particular, $\sigma_{\rm s}(T)$ contains $\partial\Xi(T)$, the topological boundary of $\Xi(T)$.*

Proof The inclusion $\Xi(T) \cup \sigma_{\rm s}(T) \subseteq \sigma(T)$ is obvious. Conversely, if $\lambda \notin \Xi(T) \cup \sigma_{\rm s}(T)$ then $\lambda I - T$ is surjective and T has the SVEP at λ, so by Corollary 2.25 $\lambda I - T$ is also injective. Hence $\lambda \notin \sigma(T)$.

The last claim is immediate, $\partial\Xi(T) \subseteq \sigma(T)$ and since $\Xi(T)$ is open it follows that $\partial\Xi(T) \cap \Xi(T) = \varnothing$. This obviously implies that $\partial\Xi(T) \subseteq \sigma_{\rm s}(T)$. ■

Corollary 2.28. *Let X be a Banach space and $T \in L(X)$. Then the following statements hold:*

(i) *If T has the SVEP then $\sigma_{\rm s}(T) = \sigma(T)$.*

(ii) *If T^* has the SVEP then $\sigma_{\rm a}(T) = \sigma(T)$.*

(iii) *If both T and T^* have the SVEP then*

$$\sigma(T) = \sigma_{\rm s}(T) = \sigma_{\rm a}(T).$$

Proof The first equality (i) is an obvious consequence of Corollary 2.27, since $\Xi(T)$ is empty. Part (ii) is obvious consequence of Theorem 2.5, while part (iii) follows from part (i) and part (ii). ■

The relative positions of many of the subspaces before introduced are intimately related to the SVEP at a point. To see that let us consider, for an arbitrary $\lambda_0 \in \mathbb{C}$ and an operator $T \in L(X)$ the following increasing chain of kernel type of spaces:

$$\ker\,(\lambda_0 I - T) \subseteq \mathcal{N}^\infty(\lambda_0 I - T) \subseteq H_0(\lambda_0 I - T) \subseteq X_T(\{\lambda_0\}),$$

and the decreasing chain of the range type of spaces:

$$X_T(\varnothing) \subseteq X_T(\mathbb{C}\setminus\{\lambda_0\}) = K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)(X).$$

Remark 2.29. In the sequel all the results stated are relative to SVEP at 0. It is clear that T has SVEP at λ_0 if and only if $S := \lambda_0 I - T$ has SVEP at 0. Therefore, all the results that we shall prove for operators T are still valid by replacing T with the operator $\lambda_0 I - T$. In this case the SVEP of T at 0 must be replaced with the SVEP of T at λ_0.

The next corollary is an immediate consequence of Theorem 2.24 and the inclusions considered above.

Corollary 2.30. *Suppose that $T \in L(X)$, X a Banach space, verifies one of the following conditions:*

(i) $\mathcal{N}^\infty(T) \cap (T)^\infty(X) = \{0\}$;

(ii) $\mathcal{N}^\infty(T) \cap K(T) = \{0\}$;

(iii) $\mathcal{N}^\infty(T) \cap X_T(\varnothing) = \{0\}$;

(iv) $H_0(T) \cap K(T) = \{0\}$;

(v) $\ker T \cap T(X) = \{0\}$.

Then T has the SVEP at 0. ■

The SVEP may be characterized as follows.

Theorem 2.31. *Let $T \in L(X)$, X a Banach space. Then T has the SVEP if and only if $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$.*

Proof Suppose first that T has the SVEP. From Theorem 2.18 we know that

$$K(\lambda I - T) = X_{\lambda I - T}(\mathbb{C}\setminus\{0\}) = X_T(\mathbb{C}\setminus\{\lambda\}) \quad \text{for every } \lambda \in \mathbb{C},$$

and, by Theorem 2.22,

$$H_0(\lambda I - T) = X_{\lambda I - T}(\{0\}) = X_T(\{\lambda\}) \quad \text{for every } \lambda \in \mathbb{C}\ .$$

Consequently by Theorem 2.14

$$H_0(\lambda I - T) \cap K(\lambda I - T) = X_T(\{\lambda\}) \cap X_T(\mathbb{C}\setminus\{\lambda\}) = X_T(\varnothing) = \{0\}\ .$$

The converse implication is clear by Corollary 2.30. ■

Corollary 2.32. *Suppose that $T \in L(X)$, X a Banach space. If T is quasi-nilpotent then $K(T) = \{0\}$.*

Proof If T is quasi-nilpotent then $H_0(T) = X$ by Theorem 2.21. On the other hand, since T has the SVEP, from Theorem 2.31 we conclude that $\{0\} = K(T) \cap H_0(T) = K(T)$. ∎

Remark 2.33. An example based on the theory of weighted shifts, shows that the SVEP at a point does not necessarily implies that $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$. See [**12**].

Theorem 2.34. *Suppose that $T \in L(X)$, where X is a Banach space, has a closed quasi-nilpotent part $H_0(T)$. Then $H_0(T) \cap K(T) = \{0\}$ and hence T has the SVEP at* 0.

Proof Assume first that $H_0(T)$ is closed. Let $\widetilde{T}$ denote the restriction of T to the Banach space $H_0(T)$. Clearly, $H_0(T) = H_0(\widetilde{T})$, so $\widetilde{T}$ is quasi-nilpotent and hence $K(\widetilde{T}) = \{0\}$, by Corollary 2.32. On the other hand it is easily seen that $H_0(T) \cap K(T) = K(\widetilde{T})$.

The last assertion is clear from Corollary 2.30. ∎

The next example shows that an operator $T \in L(X)$ may have the SVEP at the point λ_0 but fails the property of having a closed quasi-nilpotent part $H_0(\lambda_0 I - T)$.

Example 2.35. Let $X := \ell_2 \oplus \ell_2 \cdots$ provided with the norm

$$\|x\| := \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2} \qquad \text{for all } x := (x_n) \in X,$$

and define

$$T_n e_i := \begin{cases} e_{i+1} & \text{if } i = 1, \cdots, n \,, \\ \dfrac{e_{i+1}}{i-n} & \text{if } i > n. \end{cases}$$

It is easy to verify that

$$\|T_n^{n+k}\| = 1/k! \quad \text{ and } \quad (1/k!)^{1/n+k} \text{ as } k \to \infty.$$

¿From this it follows that $\sigma(T_n) = \{0\}$. Moreover, T_n is injective and the point spectrum $\sigma_{\mathrm{p}}(T_n)$ is empty, so T_n has the SVEP.

Now let us define $T := T_1 \oplus \cdots \oplus T_n \oplus \cdots$. ¿From the estimate $\|T_n\| = 1$ for every $n \in \mathbb{N}$, we easily obtain $\|T\| = 1$. Moreover, since $\sigma_{\mathrm{p}}(T_n) = \varnothing$ for every $n \in \mathbb{N}$, it also follows that $\sigma_{\mathrm{p}}(T) = \varnothing$.

Let us consider the sequence $x = (x_n) \subset X$ defined by $x_n := e_1/n$ for every n. We have

$$\|x\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty ,$$

which implies that $x \in X$. Moreover,

$$\|T^n x\|^{1/n} \geq \|T_n^n \frac{e_1}{n}\|^{1/n} = (1/n)^{1/n}$$

and the last term does not converge to 0. From this it follows that $\sigma_T(x)$ contains properly $\{0\}$ and therefore, by Theorem 2.22, $x \notin H_0(T)$. Finally,

$$\ell_2 \oplus \ell_2 \cdots \oplus \ell_2 \oplus \{0\} \cdots \subset H_0(T),$$

where the non-zero terms are n. This holds for every $n \in \mathbb{N}$, so $H_0(T)$ is dense in X. Since $H_0(T) \neq X$ it follows that $H_0(T)$ is not closed.

Theorem 2.36. *Suppose that for a bounded operator $T \in L(X)$, the sum $H_0(T) + T(X)$ is norm dense in X. Then T^* has the SVEP at 0.*

Proof From Theorem 2.23 we know that $K(T^*) \subseteq H_0(T)^\perp$. From a standard duality argument we now obtain

$$\ker (T^*) \cap K(T^*) \subseteq T(X)^\perp \cap H_0(T)^\perp = (T(X) + H_0(T))^\perp.$$

If the subspace $H_0(T) + T(X)$ is norm-dense in X, then the last annihilator is zero, so that $\ker T^* \cap K(T^*) = \{0\}$, and consequently by Theorem 2.24 T^* has the SVEP at 0. ■

Corollary 2.37. *Suppose either that $H_0(T) + K(T)$ or $\mathcal{N}^\infty(T) + (T)^\infty(X)$ is norm dense in X. Then T^* has the SVEP at 0.* ■

It is easy to find an example of an operator for which T^* has the SVEP at a point 0 and such that $\mathcal{N}^\infty(T) + T^\infty(X)$ is not norm dense in X.

Example 2.38. Let T denote the Volterra operator on the Banach space $X := C[0, 1]$ defined by

$$(Tf)(t) := \int_0^t f(s)\mathrm{d}s \quad \text{for all } f \in C[0, 1] \quad \text{and } t \in [0, 1].$$

T is injective and quasi-nilpotent. Consequently $\mathcal{N}^\infty(T) = \{0\}$ and $K(T) = \{0\}$ by Corollary 2.32. It is easy to check that

$$T^\infty(X) = \{f \in C^\infty[0, 1] : f^{(n)}(0) = 0, \ n \in \mathbb{Z}_+\},$$

thus $T^\infty(X)$ is not closed and hence is strictly larger than $K(T) = \{0\}$. Clearly the sum $\mathcal{N}^\infty(T) + T^\infty(X)$ is not norm dense in X, while T^* has the SVEP because it is quasi-nilpotent.

The following theorem establishes a first relationship between the ascent and the descent of T and the SVEP at 0.

Theorem 2.39. *For a bounded operator T on a Banach space X the following implications hold:*

$$p(T) < \infty \Rightarrow \mathcal{N}^\infty(T) \cap (T)^\infty(X) = \{0\} \Rightarrow T \textit{ has the SVEP at } 0,$$

and

$$q(T) < \infty \Rightarrow X = \mathcal{N}^\infty(T) + (T)^\infty(X) \Rightarrow T^* \textit{ has the SVEP at } 0.$$

Proof Assume that $p := p(T) < \infty$. Then $\mathcal{N}^\infty(T) = \ker T^p$, and therefore from Lemma 1.18 we obtain that

$$\mathcal{N}^\infty(T) \cap T^\infty(X) \subseteq \ker T^p \cap T^p(X) = \{0\}.$$

¿From Corollary 2.30 we then conclude that T has the SVEP at 0.

To show the second chain of implications suppose that $q := q(T) < \infty$. Then $T^\infty(X) = T^q(X)$ and

$$\mathcal{N}^\infty(T) + T^\infty(X) = \mathcal{N}^\infty(T) + T^q(X) \supseteq \ker T^q + T^q(X). \tag{24}$$

Now, the condition $q = q(T) < \infty$ yields that $T^{2q}(X) = T^q(X)$, so for every element $x \in X$ there exists $y \in T^q(X)$ such that $T^q y = T^q(x)$. Obviously $x - y \in \ker T^q$, and therefore $X = \ker T^q + T^q(X)$. ¿From the inclusion (24) we conclude that $X = \mathcal{N}^\infty(T) + T^\infty(X)$, and therefore by Corollary 2.37 T^* has the SVEP at 0. ■

As we observed in Theorem 2.39 and Theorem 2.34, each one of the two conditions $p(T) < \infty$ or $H_0(T)$ closed implies the SVEP at 0. In general these two conditions are not related. Indeed, the operator T defined in Example 2.35 has its quasi-nilpotent part $H_0(T)$ not closed whilst, being T injective, $p(T) = 0$.

In the following example we find an operator T which has a closed quasi-nilpotent part but ascent $p(T) = \infty$.

Example 2.40. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined

$$Tx := \left(\frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right), \text{ where } x = (x_n) \in \ell^2(\mathbb{N}).$$

It is easily seen that

$$\|T^k\| = \frac{1}{(k+1)!} \quad \text{for every } k = 0, 1, \dots$$

and from this it easily follows that the operator T is quasi-nilpotent and therefore $H_0(T) = \ell^2(\mathbb{N})$ by Theorem 2.21. Obviously $p(T) = \infty$.

5. The SVEP for operators of Kato type

In this section we shall characterize the SVEP at 0 in the cases of operators of Kato type introduced in the last section of Chapter 1. In fact, we shall see that most of the conditions which ensure the SVEP at a point 0, established in the previous section, are actually equivalences.

We recall first some definitions given in Chapter 1. A bounded operator $T \in L(X)$, X a Banach space, is said to be a *semi-regular* operator if T has closed range $T(X)$ and $\ker T \subseteq T^n(X)$ for every $n \in \mathbf{N}$. An operator $T \in L(X)$, X a Banach space, is said to admit a *generalized Kato decomposition*, abbreviated as GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$is semi-regular and $T|N$ is quasi-nilpotent. If $T|N$ is assumed to be nilpotent of order d then T is said to be of *Kato type of operator of order d*. An operator is said to be *essentially semi-regular* if it admits a GKD (M, N) such that N is finite-dimensional. Note that if T is essentially semi-regular then $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent. It has been observed in Theorem 1.72 that every semi-Fredholm operator is essentially semi-regular.

Note that if T is of Kato type then also T^* is of Kato type. More precisely, the pair $(N^\perp, M^\perp)$ is a GKD for T^* with $T^*|N^\perp$ semi-regular and $T^*|M^\perp$ nilpotent, see Theorem 1.43 of [**1**].

Lemma 2.41. *Suppose that T has a GKD (M, N). Then*

$$T|M \text{ is surjective } \Leftrightarrow T^*|N^\perp \text{is injective.}$$

Proof Suppose first that $T(M) = M$ and consider an arbitrary element $x^* \in \ker T^*|N^\perp = \ker T^* \cap N^\perp$. For every $m \in M$ then there exists $m' \in M$ such that $Tm' = m$. Then we have

$$x^*(m) = x^*(Tm') = (T^*x^*)(m') = 0,$$

and therefore $x^* \in M^\perp \cap N^\perp = \{0\}$.

Conversely, suppose that $T|M$ is not onto, i.e., $T(M) \subseteq M$ and $T(M) \neq M$. By assumption $T(M)$ is closed, since $T|M$ is semi-regular, and hence via the Hahn–Banach theorem there exists $z^* \in X^*$ such that $z^* \in T(M)^\perp$ and $z^* \notin M^\perp$.

Now, from the decomposition $X^* = N^\perp \oplus M^\perp$ we have $z^* = n^* + m^*$ for some $n^* \in N^\perp$ and $m^* \in M^\perp$. For every $m \in M$ we obtain

$$T^* n^*(m) = n^*(Tm) = z^*(Tm) - m^*(Tm) = 0.$$

Hence $T^* n^* \in N^\perp \cap M^\perp = \{0\}$, and therefore $0 \neq n^* \in \ker T^* \cap N^\perp$. ■

Theorem 2.42. *Suppose that $T \in L(X)$ is semi-regular. Then $\overline{H_0(T)} = \overline{\mathcal{N}^\infty(T)}$, $K(T)$ is closed and $K(T) = T^\infty(X)$. Moreover, for semi-regular operators the following equivalences hold:*

(i) *T has the SVEP at 0 precisely when T is injective or, equivalently, when T is bounded below;*

(ii) *T^* has the SVEP at 0 precisely when T is surjective.*

Proof The proof of the first equality is similar to that of part (ii) of Theorem 2.23, just to remember that if T is semi-regular then T^n is semi-regular for each $n \in \mathbb{N}$, see [**1**, Corollary 1.17]. The equality $K(T) = T^\infty(X)$ follows from Theorem 1.24. In fact the condition $\ker T^n \subseteq T(X)$ entails that $C(T) = T^\infty(X)$ and since T^n is semi-regular then $T^n(X)$ is closed for each $n \in \mathbb{N}$, hence $T^\infty(X) = C(T)$ is closed and by Theorem 1.29 it then follows that $K(T) = T^\infty(X)$.

To show part (i) we have only to prove that if T has the SVEP at 0 then T is injective. Suppose that T is not injective. The semi-regularity of T entails $T^\infty(X) = K(T)$, and $\{0\} \neq \ker T \subseteq T^\infty(X) = K(T)$, thus T does not have the SVEP at 0 by Theorem 2.24.

To show the assertion (ii) observe that if T is semi-regular then also T^* is semi-regular and we know that T is surjective if and only if T^* is bounded below. ■

The next result shows that the SVEP at 0 of a bounded operator which admits a GKD (M, N) depends essentially on the behavior of T on the first subspace M.

Theorem 2.43. *Suppose that $T \in L(X)$ admits a GKD (M, N). Then the following assertions are equivalent:*

(i) *T has the SVEP at 0;*

(ii) *$T\,|M$ has the SVEP at 0;*

(iii) *$T|M$ is injective;*

(iv) *$H_0(T) = N$;*

(v) $H_0(T)$ *is closed;*

(vi) $H_0(T) \cap K(T) = \{0\}$*;*

(vii) $H_0(T) \cap K(T)$ *is closed.*

In particular, if T *is semi-regular then the conditions* (i)–(vii) *are equivalent to the following statement:*

(viii) $H_0(T) = \{0\}$.

Proof The implication (i) $\Rightarrow$ (ii) is clear, since the SVEP at 0 of T is inherited by the restrictions on every closed invariant subspaces.

(ii) $\Rightarrow$ (iii) $T|M$ is semi-regular, so by Theorem 2.42 $T|M$ has the SVEP at 0 if and only if $T|M$ is injective.

(iii) $\Rightarrow$ (iv) If $T|M$ is injective, from Theorem 2.42 the semi-regularity of $T|M$ implies that $\overline{H_0(T|M)} = \overline{\mathcal{N}^\infty(T|M)} = \{0\}$, and hence

$$H_0(T) = H_0(T|M) \oplus H_0(T|N) = \{0\} \oplus N = N.$$

The implications (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are obvious, while the implications (v) $\Rightarrow$ (vi) and (vii) $\Rightarrow$ (i) have been proved in Theorem 2.34.

The last assertion is clear since the pair $M := X$ and $N := \{0\}$ is a GKD for every semi-regular operator. ∎

The next result shows that if the operator T admits a generalized Kato decomposition then all the implications established in the previous section are actually equivalences.

Theorem 2.44. *Suppose that* $T \in L(X)$ *admits a GKD* (M, N)*. Then the following assertions are equivalent:*

(i) T^* *has SVEP at* 0*;*

(ii) $T\ |M$ *is surjective;*

(iii) $K(T) = M$*;*

(iv) $X = H_0(T) + K(T)$*;*

(v) $H_0(T) + K(T)$ *is norm dense in* X*.*

In particular, if T *is semi-regular then the conditions* (i)–(v) *are equivalent to the following statement:*

(vi) $K(T) = X$.

Proof (i) $\Leftrightarrow$ (ii) We know that the pair $(N^\perp, M^\perp)$ is a GKD for T^*, and hence by Theorem 2.43 T^* has SVEP at 0 if and only if $T^*\ |N^\perp$ is injective. By Lemma 2.41 T^* then has the SVEP at 0 if and only if $T\ |M$ is onto.

(ii) $\Rightarrow$ (iii) If $T|M$ is surjective then $M = K(T|M) = K(T)$, by Theorem 1.71.

(iii) $\Rightarrow$ (iv) By assumption $X = M \oplus N = K(T) \oplus N$, and therefore $X = H_0(T) + K(T)$, since $N = H_0(T|N) \subseteq H_0(T)$.

The implication (iv) $\Rightarrow$ (v) is obvious, while (v) $\Rightarrow$ (i) has been established in Theorem 2.36.

The last assertion is obvious since $M := X$ and $N := \{0\}$ provides a GKD for T. ■

The next result shows that in this case to the equivalent conditions (i)–(vii) of Theorem 2.43 we can add the condition $p(T) < \infty$.

Theorem 2.45. *Let $T \in L(X)$, X a Banach space, and assume that T is of Kato type. Then the conditions* (i)–(vii) *of Theorem 2.43 are equivalent to the following assertions:*

(viii) $p(T) < \infty$;

(ix) $\mathcal{N}^\infty(T) \cap (T)^\infty(X) = \{0\}$.

In this case, if $p := p(T)$ then

$$H_0(T) = \mathcal{N}^\infty(T) = \ker T^p. \tag{25}$$

Proof Let (M, N) be a GKD for which $T|N$ is nilpotent. Assume that one of the equivalent conditions (i)–(vii) of Theorem 2.43 holds, for instance the condition $H_0(T) = N$. We also have that $\ker T^n \subseteq \mathcal{N}^\infty(T) \subseteq H_0(T)$ for every $n \in \mathbb{N}$. On the other hand, from the nilpotency of $T|N$ we know that there exists a $k \in \mathbb{N}$ for which $(T|N)^k = 0$. Therefore $H_0(T) = N \subseteq \ker T^k$ and hence $H_0(T) = \mathcal{N}^\infty(T) = \ker\, T^k$. Obviously this implies that $p(T) \leq k$, so the equivalent conditions (i)–(vii) of Theorem 2.43 imply (viii).

The implication (viii) $\Rightarrow$ (ix) has been established in Theorem 2.39 and the condition (ix) implies the SVEP at 0, again by Theorem 2.39. The equalities (25) are clear because $\ker T^k = \ker T^p$. ■

Theorem 2.46. *Let $T \in L(X)$, X a Banach space, and assume that T is of Kato type. Then the conditions* (i)–(v) *of Theorem 2.44 are equivalent to the following conditions:*

(vi) $q(T) < \infty$;

(vii) $X = \mathcal{N}^\infty(T) + (T)^\infty(X)$;

(viii) $\mathcal{N}^\infty(T) + T^\infty(X)$ *is norm dense in X.*

In this case, if $q := q(T)$ *then*

$$T^\infty(X) = K(T) = T^q(X).$$

Proof Since T is of Kato type then $K(T) = T^\infty(X)$, by Theorem 1.71. Suppose that one of the equivalent conditions (i)–(v) of Theorem 2.44 holds, in particular suppose that $K(T) = M$. Then $M = T^\infty(X) \subseteq T^n(X)$ for every $n \in \mathbb{N}$.

On the other hand, by assumption there exists a positive integer k such that $(T|N)^k = 0$, so for all $n \geq k$ we have

$$T^n(X) \subseteq T^k(X) = T^k(M) \oplus T^k(N) = T^k(M) \subseteq M,$$

and hence $T^n(X) = M$ for all $n \geq k$. Therefore $q(T) < \infty$, so (vi) is proved.

The implication (vi) ⇒ (vii) has been established in Theorem 2.39 while the implication (vii) ⇒ (viii) is obvious. Finally, by Corollary 2.37 the condition (viii) implies the SVEP at 0 for T^*, which is the condition (i) of Theorem 2.44.

The last assertion is clear since $T^q(X) = T^k(X)$. ■

In the next result we consider the case where T is essentially semi-regular, namely N is finite-dimensional and M is finite-codimensional.

Theorem 2.47. *Suppose that* $T \in L(X)$ *is a essentially semi-regular. Then the conditions* (i)–(vii) *of Theorem 2.43 and the conditions* (viii)–(ix) *of Theorem 2.45 are equivalent to the following condition:*

(a) *The quasi-nilpotent part* $H_0(T)$ *is finite-dimensional.*

In particular, if T *has the SVEP at* 0 *then* $T \in \Phi_+(X)$.

Again, the conditions (i)–(v) *of Theorem 2.44 and the conditions* (vi)– (viii) *of Theorem 2.46 are equivalent to the following condition:*

(b) *The analytical core* $K(T)$ *is finite-codimensional.*

In particular, if T^* *has the SVEP at* 0 *then* $T \in \Phi_-(X)$.

Proof The condition (iv) of Theorem 2.43 implies (a) and this implies the condition (v) of Theorem 2.43. Analogously, the condition (iii) of Theorem 2.44 implies (b) while from (b) it follows that $T^\infty(X) = K(T)$ is finite-codimensional, see Theorem 1.45. Because $T^\infty(X) \subseteq T^q(X)$ for every $q \in \mathbb{N}$ we then may conclude that $q(T) < \infty$, which is the condition (vi) of Theorem 2.46.

It remains to establish that (a) implies that $T \in \Phi_+(X)$. Clearly if $H_0(T)$ is finite- dimensional then its subspace $\ker T$ is finite-dimensional. Moreover, if (M, N) is a GKD for T such that N is finite-dimensional then $T(X) = T(M) + T(N)$ is closed since it is the sum of the closed subspace $T(M)$ and a finite-dimensional subspace of X. This shows that $T \in \Phi_+(X)$.

Analogously, from the inclusion $K(T) \subseteq T(X)$ we see that if $K(T)$ is finite-codimensional then also $T(X)$ is finite codimensional, so $T \in \Phi_-(X)$. ∎

Corollary 2.48. *Let $T \in L(X)$, X a Banach space, and suppose that $T \in \Phi_\pm(X)$. We have:*

(i) *If T has the SVEP at* 0 *then* $\operatorname{ind} T \leq 0$*;*

(ii) *If T^* has the SVEP at* 0 *then* $\operatorname{ind} T \geq 0$.

Consequently, if both the operators T and T^* have the SVEP at 0 *then T has index* 0.

Proof By Theorem 2.45 if T has the SVEP at 0 then $p(T) < \infty$, and hence $\alpha(T) \leq \beta(T)$ by part (i) of Theorem 1.21. This shows the assertion (i). The assertion (ii) follows similarly from Theorem 2.46 and part (ii) of Theorem 1.21. The last assertion is clear. ∎

Corollary 2.49. *Let $\lambda_0 \in \sigma(T)$ and assume that $\lambda_0 I - T \in \Phi_\pm(X)$. Then the following statements are equivalent:*

(i) *T and T^* have the SVEP at λ_0;*

(ii) *$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$;*

(iii) *$H_0(\lambda_0 I - T)$ is closed and $K(\lambda_0 I - T)$ is finite-codimensional;*

(iv) *λ_0 is a pole of the resolvent $(\lambda I - T)^{-1}$, equivalently* $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$*;*

(v) *λ_0 is an isolated point of $\sigma(T)$.*

In particular, if any of the equivalent conditions (i)–(v) *holds and* $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ *then*

$$H_0(\lambda_0 I - T) = \mathcal{N}^\infty(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$K(\lambda_0 I - T) = (\lambda_0 I - T)^\infty(X) = (\lambda_0 I - T)^p(X).$$

Proof The equivalences of (i), (ii), (iii), and (iv) are obtained by combining all the results previously established. The implication (iv) ⇒ (v) is obvious whilst the implication (v) ⇒ (i) is an immediate consequence of the fact that both T and T^* have the SVEP at every isolated point of the spectrum $\sigma(T) = \sigma(T^*)$. ∎

The following example shows that a Fredholm operator T having index less than 0 may be without the SVEP at 0.

Example 2.50. Let R and L denote the right shift operator and the left shift operator, respectively, on the Hilbert space $H := \ell^2(\mathbb{N})$, defined by

$$R(x) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x) := (x_2, x_3, \dots)$$

for all $(x) := (x_n) \in \ell_2(\mathbb{N})$. Clearly

$$\alpha(R) = \beta(L) = 0 \quad \text{and} \quad \alpha(L) = \beta(R) = 1,$$

so L and R are Fredholm. Let $e_n := (0, \cdots, 0, 1, 0, \dots) \in \ell^2(\mathbb{N})$, where 1 is the n-th term and all others are equal to 0. It is easily seen that $e_{n+1} \in \ker\ L^{n+1}$ while $e_{n+1} \notin \ker L^n$ for every $n \in \mathbb{N}$, so $p(L) = \infty$. Moreover, $p(R) = 0$ being R injective, and hence, since R and S are each one the adjoint of the other,

$$p(L) = q(R) = \infty \quad \text{and} \quad q(L) = p(R) = 0.$$

Consider the operator $L \oplus R \in L(H \times H)$ defined by

$$(L \oplus R)(x, y) := (Lx, Ry), \quad \text{with } x, y \in \ell_2(\mathbb{N}).$$

It is easy to verify that

$$\alpha(L \oplus R) = \alpha(L) = 1, \quad \beta(L \oplus R) = 1 \quad \text{and} \quad p(L \oplus R) = \infty.$$

Analogously, if $T := L \oplus R \oplus R \in L(H \times H \times H)$ then

$$\beta(T) = 2, \quad \alpha(T) = \alpha(L) = 1 \quad \text{and} \quad p(T) = \infty,$$

so T is a Fredholm operator having index ind $T < 0$, and, by Theorem 2.45, T does not have the SVEP at 0.

If $\lambda_0 I - T$ is of Kato type, the SVEP at λ_0 is simply characterized in terms of the approximate point spectrum as follows:

Theorem 2.51. *Suppose that $\lambda_0 I - T$ is of Kato type. Then the following statements are equivalent:*

(i) *T has SVEP at λ_0;*

(ii) *$\sigma_{\mathrm{a}}(T)$ does not cluster at λ_0.*

Proof It has been observed that if $\sigma_{\mathrm{a}}(T)$ does not cluster at λ_0 then T has the SVEP at λ_0. Hence we need only to prove the implication (i) $\Rightarrow$ (ii). We may suppose that $\lambda_0 = 0$.

Assume that T has SVEP at 0 and let (M, N) be a GKD for T. From

Theorem 1.73 we know that there exists $\varepsilon > 0$ such that $\lambda I - T$ is semi-regular, and hence has closed range for every $0 < |\lambda| < \varepsilon$. If $\mathbb{D}_\varepsilon$ denotes the open disc centered at 0 with radius ε then $\lambda \in (\mathbb{D}_\varepsilon \setminus \{0\}) \cap \sigma_{\rm a}(T)$ if and only if λ is an eigenvalue for T.

Now, from the inclusion $\ker(\lambda I - T) \subseteq T^\infty(X)$ for every $\lambda \neq 0$ we infer that every non-zero eigenvalue of T belongs to the spectrum of the restriction $T|T^\infty(X)$.

Finally, assume that 0 is a cluster point of $\sigma_{\rm a}(T)$. Let (λ_n) be a sequence of non-zero eigenvalues which converges to 0. Then $\lambda_n \in \sigma(T|T^\infty(X))$ for every $n \in \mathbb{N}$, and hence $0 \in \sigma(T|T^\infty(X))$, since the spectrum of an operator is closed. But T has the SVEP at 0, so by Theorem 2.43 the restriction $T|M$ is injective, and hence

$$\{0\} = \ker T|M = \ker T \cap K(T) = \ker T \cap T^\infty(X),$$

see Theorem 1.71. This shows that the restriction $T|T^\infty(X)$ is injective.

On the other hand, from the equality $T(T^\infty(X)) = T^\infty(X)$ we know that $T\ |T^\infty(X)$ is surjective, so $0 \notin \sigma(T\ |T^\infty(X))$; a contradiction. ■

The following result is dual to that established in Theorem 2.51.

Theorem 2.52. *Suppose that $\lambda_0 I - T$ is of Kato type. Then the following properties are equivalent:*

(i) *T^* has the SVEP at λ_0;*

(ii) *$\sigma_{\rm s}(T)$ does not cluster at λ_0.*

Proof The equivalence immediately follows from Theorem 2.51 since $\sigma_{\rm a}(T^*) = \sigma_{\rm s}(T)$. ■

By Corollary 2.25 if Y is a closed subspace of the Banach space X such that $T(Y) = Y$ and the restriction $T|Y$ has SVEP at 0 then $\ker T \cap Y = \{0\}$. The following useful result shows that this result is even true whenever we assume that Y is complete with respect to a new norm and Y is continuously embedded in X.

Lemma 2.53. *Suppose that X is a Banach space and that the operator $T \in L(X)$ has the SVEP at λ_0. Let Y be a Banach space which is continuously embedded in X and satisfies $(\lambda_0 I - T)(Y) = Y$. Then $\ker(\lambda_0 I - T) \cap Y = \{0\}$.*

Proof It follows from the closed graph theorem that the restriction $T|Y$ is continuous with respect to the given norm $\|\cdot\|_1$ on Y. Moreover, since every analytic function $f : \mathcal{U} \to (Y, \|\cdot\|_1)$ on an open set $\mathcal{U} \subseteq \mathbb{C}$

remains analytic when considered as a function from $\mathcal{U}$ to X, it is clear that $T|Y$ inherits the SVEP at λ_0 from T. Hence Corollary 2.25 applies to $T|Y$ with respect to the norm $\|\cdot\|_1$. ∎

By Theorem 2.45, if T a semi-Fredholm operator T has the SVEP at 0 precisely when $p(T) < \infty$. The next result shows that this equivalence holds also under the assumption that $q(T) < \infty$.

Theorem 2.54. *Let $T \in L(X)$, X a Banach space, and suppose that $0 < q(T) < \infty$. Then the following conditions are equivalent:*

(i) *T has the SVEP at 0;*

(ii) *$p(T) < \infty$;*

(iii) *0 is a pole of the resolvent;*

(iv) *0 is an isolated point of $\sigma(T)$.*

Proof (i) ⇒ (ii) Let $q := q(T)$ and $Y := T^q(X)$. Let us consider the map $\widehat{T} : X/\ker T^q \to Y$ defined by $\widehat{T}(\widehat{x}) := Tx$ where $x \in \widehat{x}$. Clearly, since $\widehat{T}$ is continuous and bijective we can define in Y a new norm

$$\|y\|_1 := \inf\{\|x\| : T^q(x) = y\},$$

for which $(Y, \|\cdot\|_1)$ becomes a Banach space. Moreover, if $y = T^q(x)$ from the estimate

$$\|y\| = \|T^q(x)\| \leq \|T^q\|\|x\|$$

we deduce that Y can be continuously embedded in X. Since $T(T^q(X)) = T^{q+1}(X) = T^q(X)$, by Corollary 2.53 we conclude that $\ker T \cap T^q(X) = \{0\}$ and hence by Lemma 1.18 $p(T) < \infty$.

(ii) ⇒ (iii) If $p := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$ then λ_0 is a pole of order p.

(iii) ⇒ (iv) Obvious.

(iv) ⇒ (i) This has been observed above. ∎

The preceding result is reminiscent of the equivalences established in Corollary 2.49 under the assumption that $\lambda_0 I - T$ is semi-Fredholm.

Theorem 2.55. *Suppose that $T \in L(X)$, X a Banach space, and $0 \in \sigma(T)$. Then following statements are equivalent:*

(i) *0 is a pole of the resolvent of T;*

(ii) *There exists $p \in \mathbb{N}$ such that $\ker T^p = H_0(T)$ and $T^p(X) = K(T)$.*

Proof Suppose that $0 \in \sigma(T)$ is a pole of the resolvent of T. Then $p(T)$ and $q(T)$ are finite and hence equal. Moreover, if $p := p(T) = q(T)$ then $P_0(X) = \ker T^p$ and $\ker P_0 = T^p(X)$, where P_0 is the spectral projection associated with $\{0\}$, so the assertion (ii) is true by Theorem 2.9.

Conversely, assume that (ii) is verified. We show that $p(T)$ and $q(T)$ are finite. From

$$\ker T^{p+1} \subseteq H_0(T) = \ker T^p$$

we obtain that $\ker T^{p+1} = \ker T^p$, thus $p(T) \leq p$.

¿From the inclusion

$$T^{p+1}(X) \supseteq T^{\infty}(X) \supseteq K(T) = T^p(X)$$

we then conclude that $T^{p+1}(X) = T^p(X)$, thus also $q(T)$ is finite. Therefore 0 is a pole of $R(\lambda, T)$. ■

6. Browder and Weyl operators

Two important classes of operators in Fredholm theory are given by the classes of semi-Fredholm operators which possess finite ascent or finite descent. We shall distinguish two classes of operators. The class of all *upper semi-Browder operators* on a Banach space X that is defined by

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators* that is defined by

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* (known in the literature also as *Riesz Schauder operators*) is defined by

$$\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X) = \{T \in \Phi(X) : p(T),\ q(T) < \infty\}.$$

Clearly, from part (i) and part (ii) of Theorem 1.21 we have

$$T \in \mathcal{B}_+(X) \Rightarrow \text{ind } T \leq 0,$$

and

$$T \in \mathcal{B}_-(X) \Rightarrow \text{ind } T \geq 0,$$

so that

$$T \in \mathcal{B}(X) \Rightarrow \text{ind } T = 0.$$

¿From the dual relationships between semi-Fredholm operators, ascent and descent, we also obtain that

$$T \in \mathcal{B}_+(X) \Leftrightarrow T^* \in \mathcal{B}_-(X^*)$$

and, analogously,

$$T \in \mathcal{B}_-(X) \Leftrightarrow T^* \in \mathcal{B}_+(X^*).$$

Definition 2.56. *A bounded operator $T \in L(X)$ is said to be a* Weyl operator *if T is a Fredholm operator having index* 0.

Denote by $\mathcal{W}(X)$ the class of all Weyl operators. Obviously $\mathcal{B}(X) \subseteq \mathcal{W}(X)$ and the inclusion is strict, see the operator $L \oplus R$ of Example 2.50. Combining Theorem 2.45, Theorem 2.46, and Theorem 1.21, we easily obtain for a Weyl operator T the following equivalence:

$$T \text{ has the the SVEP at } 0 \Leftrightarrow T^* \text{ has the SVEP at } 0.$$

Moreover, if T or T^* has SVEP at 0 from Theorem 1.21 we deduce that

$$T \text{ is Weyl } \Leftrightarrow T \text{ is Browder.}$$

A basic result of operator theory establishes that every finite-dimensional operator $T \in \mathcal{F}(X)$ may be always represented in the form

$$Tx = \sum_{k=1}^{n} f_k(x)x_k,$$

where the vectors $x_1, \dots, x_n$ from X and the vectors $f_1, \dots, f_n$ from X^* are linearly independent, see Heuser [**68**, p. 81]. Clearly $T(X)$ is contained in the subspace Y generated by the vectors $x_1, \dots, x_n$.

Conversely, if $y := \lambda_1 x_1 + \dots + \lambda_n x_n$ is an arbitrary element of Y we can choose $z_1, \dots, z_n$ in X such that $f_i(z_j) = \delta_{i,j}$, where $\delta_{i,j}$ denote the delta of Kronecker (a such choice is always possible, see Heuser [**68**, Proposition 15.1]). If we define $z := \sum_{k=1}^n \lambda_k z_k$ then

$$Tz = \sum_{k=1}^{n} f_k(z)x_k = \sum_{k=1}^{n} f_k\left(\sum_{k=1}^{n} \lambda_k z_k\right)x_k = \sum_{k=1}^{n} \lambda_k x_k = y.$$

This shows that the set $\{x_1, \dots, x_n\}$ forms a basis for the subspace $T(X)$.

Theorem 2.57. *For a bounded operator T on a Banach space X, the following assertions are equivalent:*

(i) *T is a Weyl operator;*

(ii) *There exist $K \in \mathcal{F}(X)$ and an invertible operator $S \in L(X)$ such that $T = S + K$ is invertible;*

(iii) *There exist $K \in \mathcal{K}(X)$ and an invertible operator $S \in L(X)$ such that $T = S + K$ is invertible.*

Proof (i) $\Rightarrow$ (ii) Assume that T is a Fredholm operator having index $\text{ind}\, T = \alpha(T) - \beta(T) = 0$ and let $m := \alpha(T) = \beta(T)$. Let $P \in L(X)$ denote the projection of X onto the finite-dimensional space $\ker T$. Obviously, $\ker T \cap \ker P = \{0\}$ and we can represent the finite-dimensional operator P in the form

$$Px = \sum_{i=1}^{m} f_i(x) x_i,$$

where the vectors $x_1, \dots, x_m$ from X, the vectors $f_1, \cdots, f_m$ from X^*, are linearly independent. As observed before, the set $\{x_1, \dots, x_m\}$ forms a basis of $P(X)$ and therefore $Px_i = x_i$ for every $i = 1, \dots, m$, from which we obtain that $f_i(x_k) = \delta_{i,k}$.

Denote by Y the topological complement of the finite-codimensional subspace $T(X)$. Then Y is finite-dimensional with dimension m, so we can choose a basis $\{y_1, \dots, y_m\}$ of Y. Let us define

$$Kx := \sum_{i=1}^{m} f_i(x) y_i$$

Clearly K is a finite-dimensional operator, so by Theorem 1.56 $S := T + K$ is a Fredholm operator and $K(X) = Y$.

Finally, consider an element $x \in \ker S$. Then $Tx = Kx = 0$, and this easily implies that $f_i(x) = 0$ for all $i = 1, \dots, m$. From this it follows that $Px = 0$ and therefore $x \in \ker T \cap \ker P = \{0\}$, so S is injective.

In order to show that S is surjective observe first that

$$f_i(Px) = f_i\left(\sum_{k=1}^{m} f_k(x) x_k\right) = f_i(x).$$

¿From this we obtain that

$$KPx = \sum_{i=1}^{m} f_i(Px) y_i = \sum_{i=1}^{m} f_i(x) y_i = Kx. \tag{26}$$

Now, we have $X = T(X) \oplus Y = T(X) \oplus K(X)$, so every $z \in X$ may be represented in the form $z = Tu + Kv$, with $u, v \in X$. Set

$$u_1 := u - Pu \quad \text{and} \quad v_1 := Pv.$$

¿From (26) and from the equality $P(X) = \ker T$ we easily obtain that

$$Ku_1 = 0, \quad Tv_1 = 0, \quad Kv_1 = Kv \quad \text{and} \quad Tu_1 = Tu.$$

Therefore

$$S(u_1 + v_1) = (T + K)(u_1 + v_1) = Tu + Kv = z,$$

and hence S is surjective. Therefore $S = T + K$ is invertible.

(ii)⇒ (iii) Clear.

(iii)⇒ (i) Suppose $T + K = U$, where U is invertible and K is compact. Obviously U is a Fredholm operator having index 0, and hence by Theorem 1.56 we conclude that $T \in \mathcal{W}(X)$. ■

Definition 2.58. *A bounded operator $T \in L(X)$ is said to be* upper semi-Weyl *if $T \in \Phi_+(X)$ and* $\text{ind}\,(T) \leq 0$. *$T \in L(X)$ is said to be* lower semi-Weyl *if $T \in \Phi_-(X)$ and* $\text{ind}\,(T) \geq 0$. *The set of all upper semi-Weyl operators will be denoted by $\mathcal{W}_+(X)$, while the set of all lower semi-Weyl operators will be denoted by $\mathcal{W}_-(X)$.*

By means of a modest modification of the proof of Theorem 2.57 we easily obtain the following characterizations of upper and lower Weyl operators::

Theorem 2.59. *Let $T \in L(X)$. Then we have*

(i) *$T \in \mathcal{W}_+(X)$ if and only if there exist $K \in \mathcal{K}(X)$ and a bounded below operator S such that $T = S + K$.*

(ii) *$T \in \mathcal{W}_-(X)$ if and only if there exist $K \in \mathcal{K}(X)$ and a surjective operator S such that $T = S + K$.*

Proof To show part (i) take $m := \alpha(T)$ and proceed as in the proof of Theorem 2.57. The operator $S = T + K$ is then injective and has closed range, since $T + K \in \Phi_+(X)$. To show part (ii) take $m := \beta(T)$ and proceeding as in the proof of Theorem 2.57. ■

Lemma 2.60. *Suppose that $T \in L(X)$ and $K \in K(X)$ commute.*

(i) *If T is bounded below then $p(T - K) < \infty$;*

(ii) *If T is onto then $q(T - K) < \infty$.*

Proof We first establish the implication (ii). The implication (i) will follows then by duality.

(ii) Obviously T is lower semi-Fredholm, and hence $S := T - K \in \Phi_-(X)$. Consequently, also $S^k \in \Phi_-(X)$ for every $k \in \mathbb{N}$, and hence the range $S^k(X)$ is finite-codimensional.

Let us consider the map $\widehat{T} : X/S^k(X) \to X/S^k(X)$, defined canonically by

$$\widehat{T}\widehat{x} := \widehat{Tx} \quad \text{for all } \widehat{x} := x + S^k(X).$$

Since T is onto, for every $y \in X$ there exists an element $z \in X$ such that $y = Tz$, and therefore $\widehat{y} = \widehat{Tz} = \widehat{T}\widehat{z}$, thus $\widehat{T}$ is onto.

Since $X/S^k(X)$ is a finite-dimensional space then $\widehat{T}$ is also injective and this easily implies that $\ker T \subseteq S^k(X)$. The surjectivity of T also implies that $\gamma(T) > 0$, $\gamma(T)$ the minimal modulus of T, and

$$\|Tx\| \geq \gamma(T)\,\mathrm{dist}(x, \ker T) \quad \text{for all } x \in X.$$

Let $z \in S^k(X)$ be arbitrarily given. The equalities

$$T(S^k(X)) = (S^kT)(X) = S^k(X)$$

show that there is some $y \in S^k(X)$ for which $Ty = z$. For every $x \in X$ we have

$$\begin{aligned} \|Tx - z\| &= \|T(x-y)\| \geq \gamma(T)\,\mathrm{dist}(x-y, \ker T) \\ &\geq \gamma(T)\,\mathrm{dist}(x-y, S^k(X)), \end{aligned}$$

where the last inequality follows from the inclusion $\ker T \subseteq S^k(X)$. Consequently, for every $x \in X$ we obtain that

$$\|Tx - z\| \geq \gamma(T)\,\mathrm{dist}(x, S^k(X)) \quad \text{for all } z \in S^k(X),$$

and this implies that

$$\mathrm{dist}(Tx, S^k(X)) \geq \gamma(T)\,\mathrm{dist}(x, S^k(X)) \quad \text{for all } k \in \mathbb{N}.$$

Suppose that $q(S) = \infty$. Then there is a bounded sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in S^n(X)$ and $\mathrm{dist}(x_n, S^{n+1}(X)) \geq 1$ for every $n \in \mathbb{N}$. For $m > n$, m and $n \in \mathbb{N}$, we have

$$Kx_m - Kx_n = (Kx_m + (T-K)x_n) - Tx_n.$$

Now

$$Kx_m \in K(S^m(X)) = S^mK(X) \subseteq S^m(X),$$

and

$$(T-K)x_n \in (T-K)^{n+1}(X) = S^{n+1}(X),$$

hence $w := Kx_m + (T-K)x_n \in S^{n+1}(X)$ for all $m > n$. Therefore

$$\begin{aligned} \|Kx_m - Kx_n\| &= \|w - Tx_n\| \geq \mathrm{dist}(Tx_n, S^{n+1}(X)) \\ &\geq \mathrm{dist}(x_n, S^{n+1}(X))\gamma(T) \geq \gamma(T), \end{aligned}$$

which contradicts the compactness of K. Therefore $S = T - K$ has finite descent.

(i) If K is a compact operator and T is bounded below then K^* is compact and T^* is onto, see Theorem 1.6. Moreover, the operators $T-K$ and T^*-K^* are semi-Fredholm, hence $p(T-K)=q(T^*-K^*)<\infty$. ■

The following result is due to Grabiner [**63**].

Theorem 2.61. *Let $T \in L(X)$, $K \in \mathcal{K}(X)$ be commuting operators on a Banach space X. Then the following equivalences hold:*

(i) *If $T \in \Phi_+(X)$ then $p(T+K)<\infty$ if and only if $p(T)<\infty$;*

(ii) *If $T \in \Phi_-(X)$ then $q(T+K)<\infty$ if and only if $q(T)<\infty$.*

Proof Suppose first that $T \in \Phi_-(X)$ and $q := q(T) < \infty$. Then $T^q(X) = T^{q+1}(X)$ and $T^q(X)$ is a closed subspace of finite-codimension, since also $T^q \in \Phi_-(X)$. Let $S := T+K$. We know by Theorem 1.50 that $S \in \Phi_-(X)$. The restriction of T to $T^q(X)$ is surjective, so by Lemma 2.60 the restriction of S to $T^q(X)$ has finite descent. ¿From this it follows that there is a positive integer k for which

$$S^m(X) \supset (S^mT^q)(X) = (S^kT^q)(X) \quad \text{for all } m \geq k.$$

We have $S^kT^q \in \Phi_-(X)$, thus the subspace $S^kT^q(X)$ has finite-codimension. From this we then conclude that $S = T+K$ has finite descent.

Conversely, assume that $q(T+K)<\infty$. Since $T+K \in \Phi_-(X)$, from the first part of the proof we obtain that $q(T) = q(T+K-K) < \infty$. Hence the equivalence (ii) is proved.

The equivalence (i) follows by duality from (ii), since T and $S = T+K$ are upper semi-Fredholm if and only if T^* and $S^* = T^*+K^*$ are lower semi-Fredholm, respectively, and hence $p(T) = q(T^*)$, $p(S) = q(S^*)$. ■

It should be noted that the equivalence (i) of Theorem 2.61 may be also deduced directly from the assertion (i) of Lemma 2.60.

We now give a characterization of semi-Browder operators by means of the SVEP.

Theorem 2.62. *For an operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) *T is essentially semi-regular and T has the SVEP at 0;*

(ii) *There exist a $K \in \mathcal{F}(X)$ and a bounded below operator $S \in L(X)$ such that $TK = KT$ and $T = S+K$ is bounded below ;*

(iii) *There exists a $K \in \mathcal{K}(X)$ and a bounded below operator $S \in L(X)$ such that $TK = KT$ and $T = S+K$;*

(iv) $T \in \mathcal{B}_+(X)$.

Proof (i) $\Rightarrow$ (ii) Suppose that T is essentially semi-regular and that T has SVEP at 0. Let (M, N) be a GKD for T, where $T|N$ is nilpotent and N is finite-dimensional. Let P denote the finite-dimensional projection of X onto N along M. Clearly P commutes with T, because N and M reduce T. Since T has the SVEP at 0 it follows that $T|M$ is injective, by Theorem 2.43 . Furthermore, the restriction $(I-T)|N$ is bijective, since from the nilpotency of $T|N$ we have $1 \notin \sigma(T|N)$. Therefore $(I-T)(N) = N$ and $\ker(I-T)|N = \{0\}$. From this it follows that

$$\begin{aligned}\ker(T-P) &= \ker(T-P)|M \oplus \ker(T-P)|N \\ &= \ker T|M \oplus \ker(I-T)|N = \{0\},\end{aligned}$$

thus the operator $T - P$ is injective. On the other hand, the equalities

$$\begin{aligned}(T-P)(X) &= (T-P)(M) \oplus (T-P)(N) \\ &= T(M) \oplus (T-I)(N) = T(M) \oplus N,\end{aligned}$$

show that the subspace $(T - P)(X)$ is closed, since it is the sum of the subspace $T(M)$, which is closed by semi-regularity, and the finite-dimensional subspace N. Therefore, the operator $T - P$ is bounded below.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Suppose that there exists a commuting compact operator K such that $T + K$ is bounded below, and therefore upper semi-Fredholm. The class $\Phi_+(X)$ is stable under compact perturbations, and hence $T + K - K = T \in \Phi_+(X)$.

On the other hand, $p(T + K) = 0$, and hence from Theorem 2.60 also $p(T) = p((T + K) - K)$ is finite.

The implication (iv) $\Rightarrow$ (i) is clear from Theorem 2.43 since every semi-Fredholm operator is essentially semi-regular and hence of Kato type, see Theorem 1.72. ∎

The next result is dual to that given in Theorem 2.62.

Theorem 2.63. *Let $T \in L(X)$, X a Banach space. Then the following properties are equivalent:*

(i) *T is essentially semi-regular and T^* has the SVEP at 0;*

(ii) *There exist a $K \in \mathcal{F}(X)$ and a surjective operator S such that $TK = KT$ and $T = S + K$;*

(iii) *There exist a $K \in \mathcal{K}(X)$ and a surjective operator S such that $TK = KT$ and $T = S + K$;*

(iv) $T \in \mathcal{B}_-(X)$.

Proof (i) $\Rightarrow$ (ii) Let T be essentially semi-regular and suppose that T^* has SVEP at 0. Let (M, N) be a GKD for T, where $T|N$ is nilpotent and N is finite-dimensional. Then $(N^\perp, M^\perp)$ is a GKD for T^*. In particular, $T^*|N^\perp$ is semi-regular.

Let P denote the finite rank projection of X onto N along M. Then P commutes with T, since N and M reduce T. Moreover, since $T^*|N^\perp$ has SVEP at 0 then $T^*|N^\perp$ is injective by Theorem 2.42, and this implies that $T|M$ is surjective, see Lemma 2.41. From the nilpotency of $T|N$ we know that the restriction $(T - I)\,|N$ is bijective, so we have

$$(T-P)(X) = (T-P)(M) \oplus (T-P)(N) = T(M) \oplus (T-I)(N) = M \oplus N = X.$$

This shows that $T + P$ is onto.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Suppose that there exists a commuting compact operator K such that $T+K$ is surjective and therefore lower semi-Fredholm. The class $\Phi_-(X)$ is stable under compact perturbations, so $(T + K) - K = T \in \Phi_-(X)$.

On the other hand, $q(T+K) = 0$, and hence, again by Lemma 2.60, also $q(T) = q((T + K) - K)$ is finite.

(iv) $\Rightarrow$ (i) This is clear from Theorem 2.44 and Theorem 1.72. ■

Combining Theorem 2.62 and Theorem 2.63 we readily obtain the following characterizations of Browder operators.

Theorem 2.64. *Let $T \in L(X)$, X a Banach space. Then the following properties are equivalent:*

(i) *T is essentially semi-regular, both T and T^* have SVEP at 0;*

(ii) *There exist $K \in \mathcal{F}(X)$ and an invertible operator S such that $TK = KT$ and $T = S + K$;*

(iii) *There exist $K \in \mathcal{K}(X)$ and an invertible operator S such that $TK = KT$ and $T = S + K$;*

(iv) *$T \in \mathcal{B}(X)$.*

The following corollary is an immediate consequence of Theorem 2.64, once observed that both the operators T and T^* have the SVEP at every $\lambda \in \partial\sigma(T)$, $\partial\sigma(T)$ the boundary of $\sigma(T)$.

Corollary 2.65. *Let $T \in L(X)$, X a Banach space, and suppose that $\lambda_0 \in \partial\sigma(T)$. Then $\lambda_0 I - T$ is essentially semi-regular if and only if*

$\lambda_0 I - T$ is semi-Fredholm, and this is the case if and only if $\lambda_0 I - T$ is Browder. ∎

For isolated points of the spectrum we have a very clear situation.

Theorem 2.66. *Let λ_0 be an isolated point of $\sigma(T)$. Then the following assertions are equivalent:*

(i) $\lambda_0 I - T \in \Phi_\pm(X)$;

(ii) $\lambda_0 I - T$ *is Browder;*

(iii) $H_0(\lambda_0 I - T)$ *is finite-dimensional;*

(iv) $K(\lambda_0 I - T)$ *is finite-codimensional.*

Proof The equivalence (i) ⇔ (ii) follows from Corollary 2.65. The implication (ii) ⇒ (iii) is clear from Theorem 2.47 since T has the SVEP at every isolated point of $\sigma(T)$. The implication (iii) ⇒ (iv) is clear, since, as observed above, $X = H_o(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$.

(iv) ⇒ (i) We have $K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)(X)$, so the finite-codimensionality of $K(\lambda_0 I - T)$ implies that also $(\lambda_0 I - T)(X)$ is finite-codimensional and hence $\lambda_0 I - T \in \Phi_-(X)$. ∎

7. Riesz operators

First we introduce another important class of operators on a Banach space X which presents some of the spectral properties of compact operators.

Definition 2.67. *A bounded operator $T \in L(X)$ on a Banach space X is said to be a* Riesz operator *if $\lambda I - T \in \Phi(X)$ for every $\lambda \in \mathbb{C} \setminus \{0\}$.*

The class of all Riesz operators will be denoted by $\mathcal{R}(X)$. The classical Riesz–Schauder theory of compact operators shows that every compact operator is Riesz. Also quasi-nilpotent operators are Riesz. Other classes of Riesz operators will be investigated in Chapter 3.

Theorem 2.68. *For a bounded operator T on a Banach space the following statements are equivalent:*

(i) T *is a Riesz operator;*

(ii) $\lambda I - T \in \mathcal{B}(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$;

(iii) $\lambda I - T \in \mathcal{W}(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$;

(iv) $\lambda I - T \in \mathcal{B}_+(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$;

(v) $\lambda I - T \in \mathcal{B}_-(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$*;*

(vi) $\lambda I - T \in \Phi_+(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$*;*

(vii) $\lambda I - T \in \Phi_-(X)$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$*;*

(viii) $\lambda I - T$ *is essentially semi-regular for all* $\lambda \in \mathbb{C} \setminus \{0\}$*;*

(ix) *Each spectral point* $\lambda \neq 0$ *is isolated and the spectral projection associated with* $\{\lambda\}$ *is finite-dimensional.*

Proof (i) $\Rightarrow$ (ii) If T is a Riesz operator the semi-Fredholm resolvent has a unique component $\mathbb{C} \setminus \{0\}$. From this it follows by Theorem 3.36 of [**1**] that both T, T^* have SVEP at every $\lambda \neq 0$. Therefore, again by Theorem 3.36 of [**1**], $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \neq 0$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are clear, so (i), (ii), and (iii) are equivalent. The implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (viii), (ii) $\Rightarrow$ (v) $\Rightarrow$ (viii) are evident, so in order to show that all these assertions are actually equivalences we need to show that (viii) $\Rightarrow$ (ii).

(viii) $\Rightarrow$ (ii) Suppose that (viii) holds. Then $\lambda I - T$ is of Kato type for all $\lambda \neq 0$, and hence, since T, T^* have the SVEP at every $\lambda \in \rho(T)$, both the operators T and T^* have the SVEP at every $\lambda \neq 0$, by Theorem 3.34 of [**1**] and Theorem 3.35 of [**1**]. From Theorem 2.64 we then conclude that $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \neq 0$.

(i) $\Rightarrow$ (ix) As above, T and T^* have the SVEP at every $\lambda \neq 0$, so by Corollary 2.49 every non-zero spectral point λ is isolated in $\sigma(T)$. From Theorem 2.66 and Theorem 2.9 it then follows that the spectral projection associated with $\{\lambda\}$ is finite-dimensional.

(ix) $\Rightarrow$ (ii) If the spectral projection associated with the spectral set $\{\lambda\}$ is finite-dimensional then $H_0(\lambda I - T)$ is finite-dimensional, so by Theorem 2.66 $\lambda I - T$ is Browder. ∎

Since every non-zero spectral point of a Riesz operator T is isolated, the spectrum $\sigma(T)$ of a Riesz operator $T \in L(X)$ is a finite set or a sequence of eigenvalues which converges to 0. Moreover, since $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$, every spectral point $\lambda \neq 0$ is a pole of $R(\lambda, T)$. Clearly, if X is an infinite-dimensional complex space the spectrum of a Riesz operator T contains at least the point 0. The following result is a consequence of the Atkinson characterization of Fredholm operators given in Corollary 1.52.

Theorem 2.69. Ruston characterization $T \in L(X)$ *is a Riesz operator if and only if* $\widehat{T} := T + K(X)$ *is a quasi-nilpotent element in the Calkin algebra* $\widehat{L} := L(X)/K(X)$.

Generally, the sum and the product of Riesz operators $T, S \in L(X)$ need not to be Riesz. However, the next result shows that is true if we assume T and S commutes.

Theorem 2.70. *If* $T, S \in L(X)$ *on a Banach space* X *the following statements hold:*

(i) *If* T *and* S *are commuting Riesz operators then* $T + S$ *is a Riesz operator;*

(ii) *If* S *commutes with the Riesz operator* T *then the products* TS *is a Riesz operator;*

(iii) *The limit of a uniformly convergent sequence of commuting Riesz operators is a Riesz operator;*

(iv) *If* T *is a Riesz operator and* $K \in K(X)$ *then* $T + K$ *is a Riesz operator.*

Proof If T, S commutes the equivalences classes $\widehat{T}, \widehat{S}$ commutes in $\widehat{L}$, so (i), (ii), and (iii) easily follow from Ruston characterization of Riesz operators and from the well known spectral radius formulas

$$r(\widehat{T} + \widehat{S}) \leq r(\widehat{T}) + r(\widehat{S}) \quad \text{and} \quad r(\widehat{T}\widehat{S}) \leq r(\widehat{T})r(\widehat{S}),$$

see RemarkerThe assertion (iv) is obvious, by the Ruston characterization of Riesz operators. ∎

It should be noted that in part (i) and part (ii) of Theorem 2.70 the assumption that T and K commute may be relaxed into the weaker assumption that T, S commute modulo $K(X)$, i.e., $TS - ST \in K(X)$.

Theorem 2.71. *Let* $T \in L(X)$, *where* X *is a Banach space, and let* f *be an analytic function on a neighborhood of* $\sigma(T)$.

(i) *If* T *is a Riesz operator and* $f(0) = 0$ *then* $f(T)$ *is a Riesz operator.*

(ii) *If* $f(T)$ *is a Riesz operator and* $f \in \mathcal{H}(\sigma(T))$ *does not vanish on* $\sigma(T) \setminus \{0\}$ *then* T *is a Riesz operator. In particular, if* T^n *is a Riesz operator for some* $n \in \mathbb{N}$ *then* T *is a Riesz operator.*

((iii) *If* M *is a closed* T*-invariant subspace of a Riesz operator* T *then the restriction* $T|M$ *is a Riesz operator.*

Proof (i) Suppose that T is a Riesz operator. Since $f(0) = 0$ there exists an analytic function g on a neighborhood of $\sigma(T)$ such that $f(\lambda) = \lambda g(\lambda)$. Hence $f(T) = Tg(T)$ and since $T, g(T)$ commute it then follows by part (ii) of Theorem 2.70 that $f(T)$ is a Riesz operator.

(ii) Assume that $f(T)$ is a Riesz operator and f vanishes only at 0. Then there exist an analytic function g on a neighborhood of $\sigma(T)$ and $n \in \mathbb{N}$ such that $f(\lambda) = \lambda^n g(\lambda)$ holds on the set of definition of f and $g(\lambda) \neq 0$. Hence $f(T) = T^n g(T)$ and $g(T)$ is invertible. The operators $f(T), g(T)^{-1}$ commute, so by part (ii) of Theorem 2.70 $T^n = f(T)g(T)^{-1}$ is a Riesz operator. Hence T^n is quasi-nilpotent modulo $K(X)$ and from this it easily follows that T is quasi-nilpotent modulo $K(X)$. By the Ruston characterization we then conclude that T is a Riesz operator.

(iii) We show first that $(\lambda I - T)(M) = M$ for all $\lambda \in \rho(T)$. The inclusion $(\lambda I - T)(M) \subseteq M$ is clear. Let $|\lambda| > r(T)$. If $R_\lambda := (\lambda I - T)^{-1}$ from the well known representation $R_\lambda = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ it follows that $R_\lambda(M) \subseteq M$. For every $x' \in M^\perp$ and $x \in M$ let us consider the analytic function $\lambda \in \rho(T) \to x'(R_\lambda x)$. This function vanishes outside the spectral disk of T, so since $\rho(T)$ is connected we infer from the identity theorem for analytic functions that $x'(R_\lambda x) = 0$ for all $\lambda \in \rho(T)$. Therefore $R_\lambda x \in M^{\perp\perp} = M$ and consequently $x = (\lambda I - T)R_\lambda x \in (\lambda I - T)(M)$. This shows that $M \subseteq (\lambda I - T)(M)$, and hence $(\lambda I - T)(M) = M$ for all $\lambda \in \rho(T)$.

Now, $\lambda I - T$ is injective for all $\lambda \in \rho(T)$, so if $\widetilde{T} := T|M$ then $\rho(T) \subseteq \rho(\widetilde{T})$ and hence $\sigma(\widetilde{T}) \subseteq \sigma(T)$. Let λ_0 be an isolated spectral point of T, and hence an isolated point of $\sigma(\widetilde{T})$. If P denotes the spectral projection associated with $\{\lambda_0\}$ and T and $\widetilde{P}$ denotes the spectral projection associated with $\{\lambda_0\}$, T and $\tilde{T}$ then, as is easy to verify, $Px = \widetilde{P}x$ for all $x \in M$. Hence $\widetilde{P}$ is the restriction of P to M, so that, since P is finite-dimensional, $\widetilde{P}$ is finite-dimensional. From part (ix) of Theorem 2.68 we then conclude that $\widetilde{T}$ is a Riesz operator. ■

Corollary 2.72. *A bounded operator T of the complex Banach spaces X is a Riesz operator if and only if T^* is a Riesz operator.*

Proof By definition, if T is a Riesz operator then $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$. Therefore $\lambda I^* - T^* \in \Phi(X^*)$ for all $\lambda \neq 0$, so T^* is Riesz. Conversely, if T^* is a Riesz operator, by what we have just proved the bi-dual T^{**} is also a Riesz operator. Since the restriction of T^{**} to the closed subspace X of X^{**} is T, it follows from part (iii) of Theorem 2.71 that T itself must be a Riesz operator. ■

If $T \in L(X)$ for any closed T-invariant subspace M of X, let $\tilde{x}$ denote the class rest $x + M$. Define $\widetilde{T}_M : X/M \to X/M$ as

$$\widetilde{T}_M \tilde{x} := \widetilde{Tx} \quad \text{for each } x \in X.$$

Evidently $\widetilde{T}_M$ is well-defined. Moreover, $\widetilde{T}_M \in L(X/M)$ since it is the composition $Q_M \widetilde{T}$, where Q_M is the canonical quotient map of X onto X/M.

Theorem 2.73. *If $T \in L(X)$ is a Riesz operator and M is a closed T-invariant subspace M of X then $\widetilde{T}_M$ is a Riesz operator.*

Proof By Corollary 2.72 T^* is a Riesz operator. The annihilator $M^\perp$ of M is a closed subspace of X^* invariant under T^*, so by part (iii) of Theorem 2.71 the restriction $T^*|M^\perp$ is a Riesz operator. Now, a standard argument shows that the dual of $\widetilde{T}_M$ may be identified with $T^*|M^\perp$, so by Corollary 2.72 we may conclude that $\widetilde{T}_M$ is a Riesz operator. ■

Further insight into the classes of Riesz operators will be given in Chapter 3.

8. Perturbations for Weyl and Browder operators

The next perturbation result is an immediate consequence of Theorem 1.52.

Theorem 2.74. *Let $T \in L(X)$ and $K \in \mathcal{K}(X)$. If T is Weyl (respectively, upper Weyl, or lower Weyl) then $T + K$ is Weyl (respectively, upper Weyl, or lower Weyl).*

The spectrum of a bounded operator T does not change if we perturb T by a commuting quasi-nilpotent operator:

Lemma 2.75. *If $T \in L(X)$ and $Q \in L(X)$ is a quasi-nilpotent operator commuting with T, then T is invertible if and only if $T + Q$ is invertible. Consequently, $\sigma(T + Q) = \sigma(T)$.*

Proof We have $T + Q = T(I + T^{-1}Q)$ and since T^{-1} and Q commute then, by Theorem 2.3,

$$r(T^{-1}Q) \le r(T^{-1})r(Q) = 0.$$

Therefore $T^{-1}Q$ is quasi-nilpotent and hence $I + T^{-1}Q$ is invertible. Since $T + Q$ is the product of two invertible operators it then follows that $T + Q$ is invertible.

Conversely, if $T+Q$ is invertible then $T = (T+Q) - Q$ is invertible. The last assertion is clear. ∎

Theorem may be improved in the case of commuting perturbations as follows.

Theorem 2.76. *Let $T \in L(X)$ and $R \in L(X)$ be a Riesz operator such that $TR = RT$.*

(i) *If $T \in \Phi(X)$ then $T + R \in \Phi(X)$ with $\text{ind}T = \text{ind}(T + R)$.*

(ii) *If $T \in \mathcal{W}(X)$ then $T + R \in \mathcal{W}(X)$.*

Proof (i) Let $\pi : L(X) \to L(X)/K(X)$ be the quotient homomorphism. By the Atkinson characterization of Fredholm operators then $\pi(T)$ is invertible, and since $\pi(R)$ is quasi-nilpotent and commutes with $\pi(T)$ then $\pi(T + R) = \pi(T) + \pi(R)$ is invertible, see Lemma 2.75, hence $T + R \in \Phi(X)$. On the other hand, if we replace R with μR, where $\mu \in [0, 1]$, then the same argument shows that $T + \mu R \in \Phi(X)$. Finally, by the continuity of the index we have $\text{ind}\, T = \text{ind}\,(T + R)$.

(ii) It is immediate from (i), ∎

The next result, due to Schechter and Whitley [**97**, Theorem 30] , extends Theorem 2.76 to semi-Fredholm operators. We shall state this result without proof, since the proof requires considerable work that would take us too far afield.

Theorem 2.77. *Let $T \in L(X)$ and $R \in L(X)$ be a Riesz operator such that $TR = RT$.*

(i) *If $T \in \Phi_+(X)$ (respectively, $T \in \Phi_-(X)$) then $T + R \in \Phi_+(X)$ (respectively, $T + R \in \Phi_-(X)$) with $\text{ind}T = \text{ind}(T + R)$.*

(ii) *If $T \in \mathcal{W}_+(X)$ then $T + R \in \mathcal{W}_+(X)$.*

(iii) *If $T \in \mathcal{W}_-(X)$ then $T + R \in \mathcal{W}_-(X)$.*

(iv) *If $T \in \mathcal{W}(X)$ then $T + R \in \mathcal{W}(X)$.*

Remark 2.78. Suppose that for a linear operator T we have $\alpha(T) < \infty$. Then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$. This may be easily seen by an inductive argument. Suppose that $\dim \ker T^n < \infty$. Since $T(\ker T^{n+1}) \subseteq \ker T^n$ then the restriction $T_0 := T|\ker T^{n+1} : \ker T^{n+1} \to \ker T^n$ has kernel equal to $\ker T$ so the canonical mapping $\hat{T} : \ker T^{n+1}/\ker T \to$

ker T^n is injective. Therefore we have dim ker $T^{n+1}/\ker T \le$ dim ker $T^n < \infty$, and since dim ker $T < \infty$ we then conclude that dim ker $T^{n+1} < \infty$.

Lemma 2.79. *If $T \in L(X)$ is invertible and $R \in L(X)$ is a Riesz operator that commutes with T then $T + R$ has finite ascent.*

Proof We have $T + R = T(I + T^{-1}R)$ and $T^{-1}R$ is a Riesz operator by part (ii) of Theorem 2.70. Therefore,

$$p(T + R) = p(T(I + T^{-1}R)) = p(I + T^{-1}R) < \infty,$$

where the last inequality follows from Theorem 2.68. ∎

The following result is due to Rakočević [**94**], but the proof here given is that simpler given by Oudghiri [**89**]. Compare this result with Theorem 2.61.

Theorem 2.80. *Let $T \in \Phi_\pm(X)$ and R a Riesz operator such that $TR = RT$. Then the following assertions hold:*

(i) *If $p(T) < \infty$ then $p(T + R) < \infty$.*

(ii) *If $q(T) < \infty$ then $q(T + R) < \infty$.*

Proof Suppose and consider first the case T injective. Then T is bounded below, in particular upper semi-Fredholm, so $S := T + R$ is upper semi-Fredholm by Theorem 2.77 with $\text{ind}\,(T + R) = \text{ind}\, T \le 0$. Since $\alpha(S) < \infty$ it then follows by Remark 2.78 that ker S^n is finite-dimensional and hence $T(\ker S^n) = \ker S^n$ for all $n \in \mathbb{N}$. By Theorem 1.29 it then follows that

$$\ker S^n \subseteq K(T) \quad \text{for all } n \in \mathbb{N}. \tag{27}$$

On the other hand, we know by Corollary 1.45 $K(T) = T^\infty(X)$ is closed. Moreover, $T(K(T)) = K(T)$ and $\ker T|K = \{0\}$, so $T|K(T)$ is invertible. By Theorem 2.71 the restriction $R|K(T)$ is a Riesz operator that commutes with $T|K(T)$, and this implies by Lemma 2.79 that $S|K(T)$ has finite ascent and consequently, by (27) S has finite ascent.

Now, let us consider the general case that $p := p(T)$ is finite. Then T has SVEP at 0 and by Theorem 2.45 and Theorem 2.47 $H_0(T) = \ker T^p$ is finite-dimensional. Consider the maps $\widehat{T}$, $\widehat{R}$, $\widehat{S}$ on $X/H_0(T)$ induced, respectively, by T, R and S. It is immediate to prove that $\widehat{T}$ is bounded below and by Theorem 2.73 $\widehat{R}$ is a Riesz operator. commuting with $\widehat{T}$. Therefore, $\widehat{S} = \widehat{T} + \widehat{R}$ is an upper semi-Fredholm operator by part (i) of Theorem 2.77 and by the first part of the proof $\hat{S}$ has finite ascent, say $k = p(\widehat{S})$. Therefore $\ker S^n \subseteq (S^k)^{-1}(H_0(T))$ for all $n \in \mathbb{N}$.

Moreover, since S is semi-Fredholm with $\operatorname{ind} S = \operatorname{ind} T < \infty$, $\ker S$ is finite-dimensional and hence coincides with $(S^k)^{-1}(H_0(T))$. Thus S has finite ascent, as desired.

(ii) Proceed by duality. ■

Corollary 2.81. *Let $T \in L(X)$ and $R \in L(X)$ be a Riesz operator commuting with T. Then we have*

(i) *$T \in \mathcal{B}_+(X)$ if and only if $T + R \in \mathcal{B}_+(X)$.*

(ii) *$T \in \mathcal{B}_-(X)$ if and only if $T + R \in \mathcal{B}_-(X)$.*

(iii) *$T \in \mathcal{B}(X)$ if and only if $T + R \in \mathcal{B}(X)$.*

Proof The implications $\Rightarrow$ in (i), (ii) and (iii) are clear by Theorem 2.80. The converse implications follow by symmetry, for instance if $T + R \in \mathcal{B}_+(X)$ then $(T + R) - R = T \in \mathcal{B}_+(X)$. ■

9. Further and recent developments

An important subspace in local spectral theory is given by the *glocal spectral subspace* $\mathcal{X}_T(F)$ associated with a closed subset $F \subseteq \mathbb{C}$. This is defined, for an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. It should be noted that T has the SVEP if and only if $\mathcal{X}_T(\Omega) = X_T(\Omega)$, for every closed subset $\Omega \subseteq \mathbb{C}$. Moreover, if $T \in L(X)$ then the quasi-nilpotent part $H_0(T) = \mathcal{X}_T(\{0\})$, see [**1**, Theorem 2.20].

The most important concept in local spectral theory is that of decomposability for operators on Banach spaces.

Definition 2.82. *Given a Banach space X, an operator $T \in L(X)$ is said to be* decomposable *if, for any open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2 = X$ and $\sigma(T|Y_k) \subseteq \mathcal{U}_k$ for $k = 1, 2$.*

A modern and more complete treatment of this class of operators may be found in the recent book of Laursen and Neumann [**76**]. This book also provides a large variety of examples and applications to several concrete cases. Decomposability of an operator may be viewed as the union of the two properties defined in the sequel. The first one, the property (β) was introduced by Bishop [**38**].

Definition 2.83. *An operator $T \in L(X)$, X a Banach space, is said to have Bishop's property (β) if, for every open set $\mathcal{U} \subseteq \mathbb{C}$ and every sequence $(f_n) \subset \mathcal{H}(\mathcal{U}, X)$ for which $(\lambda I - T)f_n(\lambda)$ converges to* 0 *uniformly on every compact subset of $\mathcal{U}$, then also $f_n \to 0$ in $\mathcal{H}(\mathcal{U}, X)$.*

The second property, which plays a remarkable role in local spectral theory, is the following one:

Definition 2.84. *An operator $T \in L(X)$, X a Banach space, is said to have the* decomposition property (δ) *if for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ we have $X = \mathcal{X}_T(\overline{\mathcal{U}_1}) + \mathcal{X}_T(\overline{\mathcal{U}_2})$.*

The following result, due to Albrecht [**25**], shows that the decomposability of an operator may be described as the conjuction of the two weaker conditions (β) and (δ).

Theorem 2.85. *For an operator $T \in L(X)$, X a Banach space, the following assertions are equivalent:*

(i) *T is decomposable;*

(ii) *T has both properties (β) and (δ);*

The properties (β) and the property (δ) are duals of each other, in the sense that an operator $T \in L(X)$, X a Banach space, has one of the properties (β) or (δ) precisely when the dual operator T^* has the other. This basic result has been established by Albrecht and Eschmeier [**26**]. Moreover, the work of Albrecht and Eschmeier gives two important characterizations of properties (β) and (δ): the property (β) characterizes the restrictions of decomposable operators to closed invariant subspaces, while the property (δ) characterizes the quotients of decomposable operators by closed invariant subspaces. The proof of this complete duality is rather complicated, since it is based on the construction of analytic functional models, for arbitrary operators defined on complex Banach spaces, in terms of certain multiplication operators defined on vector valued Sobolev type of spaces. A detailed discussion of this duality theory, together some interesting applications to the invariant problem, may be found in Chapter 2 of the monograph of Laursen and Neumann [**76**], see also the Laursen's lectures in [**46**]. Furthermore, in section 1.6 of [**76**] one can also find enlighting examples of operators T which have only the property (β) but not property (δ), and conversely T has property (δ) but not property (β).

The basic role of SVEP arises in local spectral theory since for a decomposable operator T both T and T^* satisfy SVEP. More precisely, we have ([**76**, Chapter 2]):

Theorem 2.86. *Let $T \in L(X)$ be a bounded operator on a Banach space X. The following statements hold:*

(i) *If T has the property (β) then T has the SVEP;*

(ii) *If T has the property (δ) then T^* has the SVEP;*

(iii) *T is decomposable if and only if T^* is decomposable.* ∎

Every normal operator defined on a Hilbert space, as well as every operator $T \in L(X)$ with totally disconnected spectrum, is decomposable ([**76**, Proposition 1.4.5]). In particular, every Riesz operator is decomposable. Examples of operators satisfying SVEP but not decomposable may be found among multipliers of commutative semi-simple Banach algebras, see [**76**, Chapter 4] or [**1**, Chapter 6].

We now introduce another special class of decomposable operators that will be considered later.

Definition 2.87. *An operator $T \in L(X)$, X a Banach space, is said to be* generalized scalar *if there exists a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that*

$$\Psi(1) = I \quad \textit{and} \quad \Psi(Z) = T,$$

where $\mathcal{C}^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$, and Z denotes the identity function on $\mathbb{C}$.

Every generalized scalar operator on a Banach space is decomposable, see [**76**]. In particular, every *spectral operators of finite type* is decomposable ([**42**, Theorem 3.6]).

In the previous chapter we have introduced several classes of operators for which some of the properties of semi-Fredholm operators hold. In particular, we have seen that semi-Fredholm operators are of Kato type and are quasi-Fredholm. By Theorem 1.80 in a Hilbert space every quasi-Fredholm operator is of Kato type, but it is not known if this holds in a Banach space setting. However, in the next results we shall see that most of the characterizations of localized SVEP, established in this chapter for operators of Kato type, are still valid for quasi-Fredholm operators. We first introduce the notion of Drazin invertibiliy.

Definition 2.88. *$T \in L(X)$ is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be* right Drazin invertible *if $q := q(T) < \infty$ and $T^q(X)$ is closed. A bounded operator $T \in L(X)$ is said to be* Drazin invertible *if $p(T) = q(T) < \infty$.*

It should be noted that the condition $q = q(T) < \infty$ does not entails that $T^q(X)$ is closed, see Example 5 of [**83**]. Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$.

For a proof of the following two theorems see [**5**], [**8**] and [**34**]. These theorems extend to quasi-Fredholm operators some of the characterizations of localized SVEP, established in this chapter for operators of Kato type.

Theorem 2.89. *If $T \in L(X)$, X a Banach space, is quasi-Fredholm then the following statements are equivalent:*

(i) *T has SVEP at 0;*

(ii) *$p(T) < \infty$;*

(iii) *$\sigma_{\mathrm{a}}(T)$ does not cluster at 0;*

(iv) *There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below;*

(v) *T is left Drazin invertible;*

(vi) *There exists $\nu \in \mathbb{N}$ such that $H_0(T) = \ker T^\nu$;*

(vii) *$H_0(T)$ is closed;*

(viii) *$H_0(T) \cap K(T) = \{0\}$.*

Moreover, if one of the equivalent conditions (i)–(viii) *are satisfied then T is upper semi B-Fredholm operator with index ind $T \leq 0$.*

Dually, we have:

Theorem 2.90. *If $T \in L(X)$, X a Banach space, is quasi-Fredholm then the following statements are equivalent:*

(i) *T^* has SVEP at 0;*

(ii) *$q(T) < \infty$;*

(iii) *$\sigma_{\mathrm{s}}(T)$ does not cluster at 0;*

(iv) *There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is onto;*

(v) *T is right Drazin invertible;*

(vi) $X = H_0(T) + K(T)$;

(vii) *There exists* $\nu \in \mathbb{N}$ *such that* $K(T) = T^\nu(X)$.

Moreover, if one of the equivalent conditions (i)–(vi) *are satisfied then* T *is lower semi B-Fredholm operator with index ind* $T \geq 0$.

It is natural to extend the concept of semi-Browder operators as follows:

Definition 2.91. *A bounded operator* $T \in L(X)$ *is said to be* B-Browder *(resp.* upper semi B-Browder, lower semi B-Browder*) if for some integer* $n \geq 0$ *the range* $T^n(X)$ *is closed and* $T_{[n]}$ *is Browder (resp. upper semi-Browder, lower semi-Browder). Analogously, a bounded operator* $T \in L(X)$ *is said to be* B-Weyl *(resp.* upper semi B-Weyl, lower semi B-Weyl*) if for some integer* $n \geq 0$ *the range* $T^n(X)$ *is closed and* $T_{[n]}$ *is Weyl (resp. upper semi-Weyl, lower semi-Weyl).*

There is a very clear connection between Drazin invertibilities and semi B-Browder operators, see [**31**] or [**8**]:

Theorem 2.92. *Suppose that* $T \in L(X)$. *Then the following equivalences hold:*

(i) T *is upper semi B-Browder* $\Leftrightarrow$ T *is left Drazin invertible.*

(ii) T *is lower semi B-Browder* $\Leftrightarrow$ T *is right Drazin invertible.*

(iii) T *is B-Browder* $\Leftrightarrow$ T *is Drazin invertible.*

Define

$$USBW(X) := \{T \in L(X) : T \text{ is upper semi B-Weyl}\},$$

and

$$LSBW(X) := \{T \in L(X) : T \text{ is lower semi B-Weyl}\}.$$

In the case of Hilbert space operators we have similar results to those established in Theorem 2.57 and Theorem 2.59 ([**23**]).

Theorem 2.93. *Let* $T \in L(H)$, H *a Hilbert space. Then the following statements hold.*

(i) $T \in USBW(H)$ *if and only if* $T = S + K$, *where* S *is left Drazin invertible and* $K \in \mathcal{F}(X)$.

(ii) $T \in LSBW(H)$ *if and only if* $T = S + K$, *where* S *is right Drazin invertible and* $K \in \mathcal{F}(X)$.

(iii) T *is B-Weyl if and only if* $T = S + K$, *where* S *is Drazin invertible and* $K \in \mathcal{F}(X)$.

CHAPTER 3

Perturbation classes of semi-Fredholm operators

In Fredholm theory we find two fundamentally different classes of operators: semi-groups, such as the class of Fredholm operators, the classes of upper and lower semi-Fredholm operators, and ideals of operators, such as the classes of finite-dimensional, compact operators.

A perturbation class associated with one of these semi-groups is a class of operators T for which the sum of T with an operator of the semi-group is still an element of the semi-group. A paradigm of a perturbation class is, for instance, the class of all compact operators: as we have seen in the first chapter on adding a compact operator to a Fredholm operator we obtain a Fredholm operator. For this reason the perturbation classes of operators are often called classes of *admissible* perturbations. In the first section of this chapter we shall see that the class of inessential operators is the class of *all* admissible perturbations, since it is the biggest perturbation class of the semi-group of all Fredholm operators, as well as the semi-groups of left Atkinson or right Atkinson operators. These results will be established in the general framework of operators acting between two different Banach spaces. In the first section the class of inessential operators $\mathcal{I}(X,Y)$ presents also an elegant duality theory. Indeed, the class $\mathcal{I}(X,Y)$ may be characterized by means of th defect α as well as the defect β. In the other section we shall introduce some other classes of operators, as Ω_+ and Ω_--operators, strictly singular and strictly cosingular operators. The classes $\Omega_+(X)$ and $\Omega_-(X)$, which are in a sense the dual of each other, have been introduced in Aiena [**2**] and [**3**] in order to give an internal characterization of Riesz and inessential operators.

1. Inessential operators

Given a Banach algebra $\mathcal{A}$ with unit element $e \neq 0$ the *radical* of$\mathcal{A}$ is defined by

$$\operatorname{rad} \mathcal{A} := \{r \in \mathcal{A} : e - ar \text{ is invertible for every } a \in \mathcal{A}\}.$$

Note that the radical is a two-sided ideal of $\mathcal{A}$, see Bonsall and Duncan [**39**].

Now, suppose that X is an infinite-dimensional Banach space, so that the Calkin algebra $\widehat{L} := L(X)/\mathcal{K}(X)$ is an algebra with a non-zero unit element. Consider the radical of $\widehat{L}$,

$$\operatorname{rad}\widehat{L} := \{R \in \widehat{L} : \widehat{I} - \widehat{T}\widehat{R} \text{ is invertible for every } \widehat{T} \in \widehat{L}\}.$$

The following class of operators has been introduced by Kleinecke [**70**]

Definition 3.1. *The class of all* inessential operators *on a complex infinite-dimensional Banach space X is defined:*

$$\mathcal{I}(X) := \pi^{-1}(rad\ (L(X)/\mathcal{K}(X)).$$

where $\pi : L(X) \to L(X)/\mathcal{K}(X)$ denotes the canonical quotient mapping.

Clearly, $\mathcal{I}(X)$ is a closed ideal of $L(X)$, since the radical of a Banach algebra is closed [**39**, p. 124] . By the Atkinson characterization of Fredholm operators, $T \in \Phi(X)$ if and only if $\widehat{T}$ is invertible in $\widehat{L}$, so that

$$\begin{aligned} (28)\quad \mathcal{I}(X) &= \{T \in L(X) : I - ST \in \Phi(X) \text{ for all } S \in L(X)\} \\ &= \{T \in L(X) : I - TS \in \Phi(X) \text{ for all } S \in L(X)\}. \end{aligned}$$

Taking in (28) $S = 1/\lambda I$, with $\lambda \neq 0$, we obtain that $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$, so every inessential operator is a Riesz operator.

The characterization (28) suggests how to extend the concept of inessential endomorphisms to operators acting between different Banach spaces:

Definition 3.2. *A bounded operator $T \in L(X,Y)$, where X and Y are Banach spaces, is said to be an* inessential operator *if $I_X - ST \in \Phi(X)$ for all $S \in L(Y,X)$.*

The class of all inessential operators is denoted by $\mathcal{I}(X,Y)$.

Theorem 3.3. *$\mathcal{I}(X,Y)$ is a closed subspace of $L(X,Y)$ which contains $\mathcal{K}(X,Y)$. Moreover, if $T \in I(X,Y)$, $R_1 \in L(Y,Z)$, and $R_2 \in L(W,X)$, where X, Y and W are Banach spaces, then $R_1TR_2 \in \mathcal{I}(W,Z)$.*

Proof To show that $\mathcal{I}(X,Y)$ is a subspace of $L(X,Y)$ let $T_1, T_2 \in \mathcal{I}(X,Y)$. Then, given $S \in L(Y,X)$, we have $I_X - ST_1 \in \Phi(X)$ and hence by the Atkinson characterization of Fredholm operators $I_X - ST_1$

is invertible in $L(X)$ modulo $\mathcal{K}(X)$. Therefore there exist operators $U_1 \in L(X)$ and $K_1 \in \mathcal{K}(X)$ such that

$$U_1(I_X - ST_1) = I_X - K_1.$$

From the definition of inessential operators we also have $I_X - U_1ST_2 \in \Phi(X)$, so there are $U_2 \in L(X)$ and $K_2 \in \mathcal{K}(X)$ such that

$$U_2(I_X - U_1ST_2) = I_X - K_2.$$

Then

$$U_2U_1[I_X - S(T_1 + T_2)] = U_2(I_X - K_1 - U_1ST_2) = I_X - K_2 - U_2K_1.$$

Since $K_2 + U_2K_1 \in \mathcal{K}(X)$ this shows that $I_X - S(T_1 + T_2) \in \Phi(X)$ for all $S \in L(Y, X)$. Thus $T_1 + T_2 \in \mathcal{I}(X, Y)$ and hence $\mathcal{I}(X, Y)$ is a linear subspace of $L(X, Y)$.

The property of $\mathcal{I}(X, Y)$ being closed is a consequence of $\Phi(X, Y)$ being an open subset of $L(X, Y)$ (see the proof of next Theorem 3.14). Moreover, $\mathcal{K}(X, Y) \subseteq \mathcal{I}(X, Y)$, since if $T \in \mathcal{K}(X, Y)$ then $ST \in \mathcal{K}(X)$ for every $S \in L(Y, X)$, and hence $I_X - ST \in \Phi(X)$.

To show the last assertion suppose that $T \in L(X, Y)$, $R_1 \in L(Y, Z)$ and $R_2 \in L(W, X)$. We need to prove that $I_W - SR_1TR_2 \in \Phi(W)$ for all $S \in L(Z, W)$. Given $S \in L(Z, W)$, as above, there exist $U \in L(X)$ and $K \in \mathcal{K}(X)$ with

$$U(I_X - R_2SR_1T) = I_X - K.$$

Define

$$U_0 := I_W + SR_1TUR_2 \quad \text{and} \quad K_0 := SR_1TKR_2.$$

Then $K_0 \in \mathcal{K}(W)$ and

$$\begin{aligned} U_0(I_W - SR_1TR_2) &= I_W - SR_1TR_2 + SR_1TU(I_X - R_2SR_1T)R_2 \\ &= I_W - SR_1TR_2 + SR_1T(I_X - K)R_2 \\ &= I_W - SR_1TKR_2 = I_W - K_0. \end{aligned}$$

Therefore $I_W - SR_1TR_2 \in \Phi(W)$ for all $S \in L(Z, W)$. This shows that $R_1TR_2 \in \mathcal{I}(W, Z)$. ■

If $T \in L(X, Y)$ we define by $\overline{\beta}(T)$ the codimension of the closure of $T(X)$. Clearly $\overline{\beta}(T) \leq \beta(T)$, and if $\beta(T)$ is finite then $\overline{\beta}(T) = \beta(T)$ since $T(X)$ is closed by Corollary 1.8.

Lemma 3.4. *If* $T \in L(X,Y)$ *and* $S \in L(Y,X)$ *then the following equalities hold:*

(i) $\alpha(I_X - ST) = \alpha(I_Y - TS)$;

(ii) $\beta(I_X - ST) = \beta(I_Y - TS)$;

(iii) $\overline{\beta}(I_X - ST) = \overline{\beta}(I_Y - TS)$.

Proof Obviously $T(\ker(I_X - ST)) \subseteq \ker(I_Y - TS)$ and the induced operator $\widetilde{T} : \ker(I_X - ST) \to \ker(I_Y - TS)$ is invertible, with inverse induced by S. This proves (i). There is also the inclusion

$$T(I_X - ST)(X) \subseteq (I_Y - TS)(Y)$$

and the induced operator

$$\check{T} : X/(I_X - ST)(X) \to Y/(I_Y - TS)(Y)$$

is invertible with inverse induced by S. This proves (ii), and the same argument, replacing ranges by their closure, proves (iii). The last assertion is obvious. ∎

Corollary 3.5. $T \in \mathcal{I}(X,Y)$ *if and only if* $I_Y - TS \in \Phi(Y)$.

Proof Clear. ∎

Corollary 3.6. *If* $T^* \in \mathcal{I}(Y^*, X^*)$ *then* $T \in \mathcal{I}(X,Y)$.

Proof Suppose that $T \notin \mathcal{I}(X,Y)$. Then by definition there exists an operator $S \in L(Y,X)$ such that $I_X - ST \notin \Phi(X)$. By duality this implies that ${I_X}^* - T^*S^* \notin \Phi(X^*)$ and hence $T^* \notin \mathcal{I}(Y^*, X^*)$, by Corollary 3.5. ∎

The class of inessential operators $\mathcal{I}(X,Y)$ presents a perfect symmetry with respect to the defects α and β. We prove first that the inessential operators may be characterized only by means of the nullity α.

Theorem 3.7. *For a bounded operator* $T \in L(X,Y)$, *where* X *and* Y *are Banach spaces, the following assertions are equivalent:*

(i) T *is inessential;*

(ii) $\alpha(I_X - ST) < \infty$ *for all* $S \in L(Y,X)$;

(iii) $\alpha(I_Y - TS) < \infty$ *for all* $S \in L(Y,X)$.

Proof The implication (i) $\Rightarrow$ (ii) is obvious by Corollary 3.5, whilst the equivalence of (ii) and (iii) follows from Lemma 3.4. We prove the implication (ii) $\Rightarrow$ (i).

Assume that the assertion (ii) holds. We claim that $I_X - ST \in \Phi_+(X)$ for all $S \in L(Y, X)$. Suppose that this is not true. Then there is an operator $S_1 \in L(Y, X)$ such that $I_X - S_1T \notin \Phi_+(X)$. By part (i) of Theorem 1.57 there is then a compact operator $K \in L(X)$ such that $\alpha(I_X - S_1T - K) = \infty$. Let $M := \ker\,(I_X - S_1T - K)$ and denote by J_M the embedding map from M into X. From $(I_X - S_1T)|M = K|M$ it follows that

$$KJ_M = (I_X - S_1T)J_M. \tag{29}$$

But $I_X - K \in \Phi(X)$, so by the Atkinson characterization of Fredholm operators the exists $U \in L(X)$ and a finite-dimensional operator $P \in L(X)$ such that

$$U(I_X - K) = I_X - P. \tag{30}$$

Multiplying the left side of (29) by U we obtain

$$UKJ_M = U(I_X - S_1T)J_M = UJ_M - US_1TJ_M,$$

which yields

$$US_1TJ_M = UJ_M - UKJ_M = U(I_X - K)J_M.$$

From the equality (30) we obtain $US_1TJ_M = (I_X - P)J_M$ and hence

$$(I_X - US_1T)J_M = PJ_M.$$

Note that PJ_M is a finite-dimensional operator, so $\alpha(PJ_M) = \infty$. Clearly,

$$\alpha(I_X - US_1T) = \alpha(I_X - US_1T)J_M = \alpha(PJ_M) = \infty,$$

so, for $S := US_1$ we have $\alpha(I_X - ST) = \infty$, contradicting our hypothesis (ii) . Therefore our claim is proved.

In particular, $\lambda I_X - ST \in \Phi_+(X)$ for all non-zero $\lambda \in \mathbb{C}$, so ST is a Riesz operator by Theorem 2.68, and this implies that $I_X - ST \in \Phi(X)$. Therefore T is inessential. ∎

Dually $\mathcal{I}(X, Y)$ may be characterized only by means of the deficiencies $\beta(T)$ and $\overline{\beta}(T)$.

Theorem 3.8. *For a bounded operator $T \in L(X, Y)$, where X and Y are Banach spaces, the following assertions are equivalent:*

(i) *$T \in \mathcal{I}(X, Y)$;*

(ii) $\overline{\beta}(I_X - ST) < \infty$ *for all* $S \in L(Y, X)$;

(iii) $\overline{\beta}(I_Y - TS) < \infty$ *for all* $S \in L(Y, X)$.

Proof (i) $\Rightarrow$ (ii) Since $\overline{\beta}(I_X - ST) \leq \beta(I_X - ST)$, if $T \in \mathcal{I}(X, Y)$ then $\overline{\beta}(I_X - ST) < \infty$. The implication (ii) $\Rightarrow$ (iii) is clear from Lemma 3.4.

To prove the implication (iii) $\Rightarrow$ (i) suppose that $\overline{\beta}(I_Y - TS) < \infty$ for all $S \in L(Y, X)$. We show that $I_Y - TS \in \Phi_-(Y)$ for all $S \in L(Y, X)$. Suppose that this is not true. Then there exists an operator $S_1 \in L(Y, X)$ such that $(I_Y - TS_1) \notin \Phi_-(Y)$. By part (ii) of Theorem 1.57 there is then a compact operator $K \in L(Y)$ such that $\overline{\beta}(I_Y - TS_1 - K) = \infty$. Let $N := \overline{(I_Y - TS_1 - K)(Y)}$ and denote by Q_N the canonical quotient map from Y onto Y/N. Then

$$Q_N K = Q_N(I_Y - TS_1), \tag{31}$$

and since $I_Y - K \in \Phi(Y)$ by the Atkinson characterization of Fredholm operators there exists some $U \in L(Y)$ and a finite-dimensional operator $P \in L(Y)$ such that

$$(I_Y - K)U = I_Y - P. \tag{32}$$

Multiplying the right hand side of (31) by U we obtain

$$Q_N KU = Q_N(I_Y - TS_1)U = Q_N U - Q_N TS_1 U,$$

which implies

$$Q_N TS_1 U = Q_N U - Q_N KU = Q_N(I_Y - K)U.$$

From the equality (32) we have $Q_N TS_1 U = Q_N(I_Y - P)$, and consequently

$$Q_N(I_Y - TS_1 U) = Q_N P.$$

The operator $Q_N P$ is finite-dimensional operator, so its range is closed, and therefore

$$Q_N(I_Y - TS_1 U)(Y) = \overline{Q_N(I_Y - TS_1 U)(Y)}.$$

Now,

$$\begin{aligned} \operatorname{codim} \overline{(I_Y - TS_1 U)(Y)} &\geq \operatorname{codim} \overline{Q_N(I_Y - TS_1 U)(Y)} \\ &= \operatorname{codim} Q_N(I_Y - TS_1 U)(Y) \\ &= \operatorname{codim} (Q_N P)(Y) = \infty, \end{aligned}$$

and hence $\overline{\beta}(I_Y - TS_1 U) = \infty$, contradicting our hypothesis. Therefore $I_Y - TS \in \Phi_-(Y)$ for all $S \in L(Y, X)$. In particular, $\lambda I_Y - TS \in \Phi_-(Y)$ for all non-zero $\lambda \in \mathbb{C}$, so TS is a Riesz operator by Theorem 2.68, and this implies that $I_Y - ST \in \Phi(Y)$. Therefore T is inessential by Corollary 3.5. ■

Corollary 3.9. *For $T \in L(X, Y)$, where X and Y are Banach spaces, the following statements are equivalent:*

(i) *$T \in \mathcal{I}(X, Y)$;*

(ii) *$\beta(I_X - ST) < \infty$ for all $S \in L(Y, X)$;*

(iii) *$\beta(I_Y - TS) < \infty$ for all $S \in L(Y, X)$.*

Proof (i) $\Leftrightarrow$ (ii) If $T \in I(X, Y)$ then $\beta(I_X - ST) < \infty$ for all $S \in L(Y, X)$. Conversely, if $\beta(I_X - ST) < \infty$ for all $S \in L(Y, X)$, from $\overline{\beta}(I_X - ST) \leq \beta(I_X - ST)$ and Theorem 3.8 we deduce that $T \in \mathcal{I}(X, Y)$.

The equivalence (i) $\Leftrightarrow$ (iii) is obvious by Lemma 3.4. ■

It should be noted that if X is an infinite-dimensional Banach space then $\mathcal{I}(X) \neq L(X)$ since the operator identity is not essential. The next examples show that the class of all inessential operators $\mathcal{I}(X, Y)$ may coincide with the class of *all* bounded linear operators $L(X, Y)$. This fact contrasts, curiously, with the historical denomination given by Kleinecke [**70**] at this class.

Example 3.10. $\mathcal{I}(X, Y) = L(X, Y)$ if X contains no copies of c_0 and $Y = C(K)$. $\mathcal{I}(X, Y) = L(X, Y)$ if X contains no copies of ℓ^∞ and $Y = \ell^\infty$, $H^\infty(\mathbb{D})$, or a $C(K)$, with K σ-Stonian, see [**1**, Chapter 7].

Example 3.11. $\mathcal{I}(X, Y) = L(X, Y)$ whenever X or Y are ℓ^p, with $1 \leq p \leq \infty$, or c_0, and X, Y are different. In fact, for $1 \leq p < q < \infty$ every operator from ℓ^q, or c_0, into ℓ^p is compact, see Lindenstrauss and Tzafriri [**74**, 2.c.3]. The case $p = \infty$ is covered by Example 3.10.

Other examples of Banach spaces for which $\mathcal{I}(X, Y) = L(X, Y)$ may be found in [**1**, Chapter 7]. Note that the property of being $\mathcal{I}(X, Y) = L(X, Y)$ is symmetric:

Theorem 3.12. *If X and Y are two Banach spaces then $L(X, Y) = \mathcal{I}(X, Y)$ if and only if $L(Y, X) = \mathcal{I}(Y, X)$.* ■

Proof This is an immediate consequence of Lemma 3.4.

2. Perturbation classes

We now see that the inessential operators may be characterized as perturbation classes. In the sequel, given two Banach spaces X and Y, by $\Sigma(X,Y)$ we shall denote any of the semi-groups $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_{\mathrm{l}}(X,Y)$ or $\Phi_{\mathrm{r}}(X,Y)$.

Definition 3.13. *If* $\Sigma(X,Y) \neq \varnothing$ *the* perturbation class $P\Sigma(X,Y)$ *is the set defined by*

$$P\Sigma(X,Y) := \{T \in L(X,Y) : T + \Sigma(X,Y) \subseteq \Sigma(X,Y)\}.$$

Note that $P\Sigma(X) := P\Sigma(X,X)$ is always defined since $I \in \Sigma(X) := \Sigma(X,X)$.

Theorem 3.14. *Given two Banach spaces* X, Y *for which* $\Sigma(X,Y) \neq \varnothing$, *the following assertions hold:*

(i) $P\Sigma(X,Y)$ *is a closed linear subspace of* $L(X,Y)$.

(ii) *If* $T \in P\Sigma(X,Y)$, *then both* TS *and* UT *belong to* $P\Sigma(X,Y)$ *for every* $S \in L(X)$ *and* $U \in L(Y)$.

(iii) $P\Sigma(X)$ *is a closed two-sided ideal of* $L(X)$.

Proof We show the assertions (i), (ii) and (iii) in the case $\Sigma(X,Y) = \Phi(X,Y)$. The proof in the other cases is similar.

(i) Clearly $P\Phi(X,Y)$ is a linear subspace of $L(X,Y)$. To show that $P\Phi(X,Y)$ is closed assume that $T_n \in P\Phi(X,Y)$ converges to $T \in L(X,Y)$. Given $S \in \Phi(X,Y)$ there exists $\varepsilon > 0$ so that $S+K \in \Phi(X,Y)$ for all $\|K\| < \varepsilon$, see Theorem 1.60. Now, writing

$$T + S = T_n + (T - T_n) + S$$

we see that $T + S \in \Phi(X,Y)$. Therefore $T \in P\Phi(X,Y)$.

(ii) Let $T \in P\Phi(X,Y)$ and $U \in L(Y)$. Assume first that U is invertible. For every $S \in \Phi(X,Y)$ it then follows that

$$S + UT = U(U^{-1}S + T) \in \Phi(X,Y).$$

Consequently $UT \in P\Phi(X,Y)$. The general case that U is not invertible may be reduced to the previous case, since every operator U is the sum of two invertible operators $U = \lambda I_Y + (U - \lambda I_Y)$, with $0 \neq \lambda \in \rho(U)$. A similar argument shows that $TS \in P\Phi(X,Y)$ for every $S \in L(X)$.

(iii) Clear. ■

We now characterize the inessential operators as the perturbation class of Fredholm operators in the case of operators acting on a single Banach space.

Theorem 3.15. *For every Banach space* X *we have*

$$\mathcal{I}(X) = P\Phi(X) = P\Phi_{\mathrm{l}}(X) = P\Phi_{\mathrm{r}}(X).$$

Proof We only prove the equality $\mathcal{I}(X) = P\Phi(X)$. Let $T \in \mathcal{I}(X)$ and S be any operator in $\Phi(X)$. We show that $T + S \in \Phi(X)$. Since $S \in \Phi(X)$ there exist by the Atkinson characterization of Fredholm operators some operators $U \in L(X)$ and $K \in \mathcal{K}(X)$ such that $SU = I + K$. Since $T \in \mathcal{I}(X)$ it then follows that $I + UT \in \Phi(X)$ and hence also $S(I + UT) \in \Phi(X)$, so the residual class

$$S(\widehat{I + UT}) := S(I + UT) + \mathcal{K}(X)$$

is invertible in the quotient algebra $L(X)/\mathcal{K}(X)$, by the Atkinson characterization of Fredholm operators.

On the other hand, since $KT \in \mathcal{K}(X)$ then

$$S(\widehat{I + UT}) = \widehat{S} + \widehat{SUT} = \widehat{S} + (\widehat{I + K})T = \widehat{S} + \widehat{T} + \widehat{KT} = \widehat{S + T},$$

and hence $\widehat{S + T}$ is invertible. Therefore $S+T \in \Phi(X)$, and hence, since S is arbitrary, $T \in P\Phi(X)$.

To show the opposite implication suppose that $T \in P\Phi(X)$, namely $T + \Phi(X) \subseteq \Phi(X)$. We show first that $I - ST \in \Phi(X)$ for all $S \in \Phi(X)$.

If $S \in \Phi(X)$ there exists $U \in L(X)$ and $K \in \mathcal{K}(X)$ such that $SU = I - K$. Since $U \in \Phi(X)$, by Theorem 1.46 and from the assumption we infer that $U - T \in \Phi(X)$, and hence also

$$S(U - T) = SU - ST = I - K - ST \in \Phi(X).$$

Now, adding K to $I - K - ST$ we conclude that $I - ST \in \Phi(X)$, as claimed.
To conclude the proof let $W \in L(X)$ be arbitrary. Write $W = \lambda I + (W - \lambda I)$, with $0 \neq \lambda \in \rho(T)$ then

$$I - WT = I - (\lambda I + (W - \lambda I))T = -\lambda T + (I - (\lambda I - W)T)$$

and, since $\lambda I - W \in \Phi(X)$, by Theorem 3.14, part (ii) we have $(\lambda I - W)T \in P\Phi(X)$, and hence $I + (\lambda I - W)T \in \Phi(X)$, so

$$I - WT \in -\lambda T + \Phi(X) \subseteq \Phi(X),$$

which completes the proof. ∎

We now extend the result of Theorem 3.15 to inessential operators acting between different Banach spaces.

Theorem 3.16. *Let $\Sigma(X,Y)$ denote one of the semi-groups $\Phi(X,Y)$, $\Phi_{\mathrm{l}}(X,Y)$, and $\Phi_{\mathrm{r}}(X,Y)$. If $\Sigma(X,Y) \neq \varnothing$ then $P\Sigma(X,Y) = \mathcal{I}(X,Y)$.*

Proof Also here we prove only the equality $P\Phi(X,Y) = \mathcal{I}(X,Y)$. The other cases are analogous.

Let $S \in \Phi(X,Y)$ and $T \in \mathcal{I}(X,Y)$. Take $U \in L(Y,X)$ and $K \in K(X,Y)$ such that $US = I_X - K$. Note that $U \in \Phi(Y,X)$ by Theorem 1.46. Then $US \in \Phi(X)$, and since $UT \in \mathcal{I}(X)$ from Theorem 3.14 we obtain

$$U(T+S) = UT + US \in \Phi(X).$$

From this it then follows that $T + S \in \Phi(X,Y)$.

For the inverse inclusion assume that $T \in L(X,Y)$ and $T \notin \mathcal{I}(X,Y)$. Then there exists $S \in L(Y,X)$ such that $I_X - ST \notin \Phi(X)$. Taking any operator $V \in \Phi(X,Y)$ we have

$$V(I_X - ST) = V - VST \notin \Phi(X,Y),$$

so $VST \notin P\Sigma(X,Y)$. This implies by part (ii) of Theorem 3.14 that $T \notin P\Sigma(X,Y)$, which concludes the proof. ■

Also in the characterization of $\mathcal{I}(X,Y)$ established in Theorem 3.16 we may only consider one of the two defects α or β.

Theorem 3.17. *If $\Phi(X,Y) \neq \varnothing$, for every $T \in L(X,Y)$ the following assertions are equivalent:*

(i) $T \in \mathcal{I}(X,Y)$;

(ii) $\alpha(T+S) < \infty$ *for all* $S \in \Phi(X,Y)$;

(iii) $\overline{\beta}(T+S) < \infty$ *for all* $S \in \Phi(X,Y)$;

(iv) $\beta(T+S) < \infty$ *for all* $S \in \Phi(X,Y)$.

Proof (i) $\Leftrightarrow$ (ii) Let $T \in \mathcal{I}(X,Y)$ and suppose that $S \in L(X,Y)$ is any Fredholm operator. By Theorem 3.16 it follows that $T + S \in \Phi(X,Y)$ and therefore $\alpha(T+S) < \infty$. Conversely, suppose that $\alpha(T+S) < \infty$ for each $S \in \Phi(X,Y)$. Since $\mathcal{K}(X,Y) \subseteq \mathcal{I}(X,Y)$ then, always by Theorem 3.16,

$$\frac{1}{\lambda}(S-K) \in \Phi(X,Y) \quad \text{for all } \mathcal{K} \in K(X,Y) \quad \text{and } \lambda \neq 0.$$

Therefore

$$\alpha\left(T + \frac{S-K}{\lambda}\right) = \alpha(\lambda T + S - K) < \infty \quad \text{for all } K \in \mathcal{K}(X,Y).$$

By Theorem 1.57 it follows that $\lambda T - S \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C}$. In particular, this is true for any $|\lambda| \leq 1$. Now, if $\beta(T+S)$ were infinite we would have ind $(T+S) = -\infty$, and using the stability of the index we would have

$$\text{ind }(\lambda T + S) = -\infty \quad \text{for each } |\lambda| \le 1.$$

Therefore $\beta(S) = \infty$, contradicting the assumption that $S \in \Phi(X,Y)$.

Therefore $\beta(T+S) < \infty$, and consequently $T+S \in \Phi(X,Y)$. By Theorem 3.16 we then conclude that $T \in \mathcal{I}(X,Y)$.

(i) $\Leftrightarrow$ (iii) We use an argument dual to that used in the proof of the equivalence (i) $\Leftrightarrow$ (ii). Let $T \in \mathcal{I}(X,Y)$ and $S \in \Phi(X,Y)$. By Theorem 3.16 it follows that $\beta(T+S) = \overline{\beta}(T+S) < \infty$.

Conversely, suppose that $\overline{\beta}(T+S) < \infty$ for each $S \in \Phi(X,Y)$. As above, from the inclusion $\mathcal{K}(X,Y) \subseteq \mathcal{I}(X,Y)$ we obtain for all $\lambda \neq 0$ that

$$\overline{\beta}\left(T + \frac{S-K}{\lambda}\right) = \overline{\beta}(\lambda T + S - K) < \infty \quad \text{for all } K \in \mathcal{K}(X,Y).$$

¿From Theorem 1.57 it follows that $\lambda T + S \in \Phi_-(X,Y)$ for all $\lambda \in \mathbb{C}$. In particular, this is true for any $|\lambda| \le 1$. Now, if $\alpha(T+S)$ were infinite, we would have ind $(T+S) = \infty$, and from the stability of the index we would have

$$\text{ind }(\lambda T + S) = \infty \quad \text{for each } |\lambda| \le 1.$$

Hence $\alpha(S) = \infty$, contradicting the assumption that $S \in \Phi(X,Y)$. Therefore $\alpha(T+S) < \infty$, and consequently $T+S \in \Phi(X,Y)$. By Theorem 3.16 we then deduce that $T \in \mathcal{I}(X,Y)$.

Clearly (i) $\Rightarrow$ (iv) follows from Theorem 3.16, whilst the implication (iv) $\Rightarrow$ (iii) is obvious; so the proof is complete. ■

3. The classes Ω_+ and Ω_-

Let us consider the following two classes of operators on a Banach space X introduced by Aiena in [**3**]:

$$\Omega_+(X) := \left\{ T \in L(X) : \begin{array}{l} T|M \text{ is an (into) isomorphism for no infinite-} \\ \text{dimensional, invariant subspace } M \text{ of } T, \end{array} \right\}$$

and

$$\Omega_-(X) := \left\{ T \in L(X) : \begin{array}{l} Q_M T \text{ is surjective for no infinite-} \\ \text{codimensional, invariant subspace } M \text{ of } T. \end{array} \right\}$$

Observe that if M is a closed T-invariant subspace of a Banach space X and $\widetilde{x} := x + M$ is any element in the quotient space X/M, then the induced canonical map $T^M : X/M \to X/M$, defined by

$$T^M \widetilde{x} := \widetilde{Tx} \quad \text{for every } x \in X,$$

has the same range of the composition map $Q_M T$.

Theorem 3.18. *If T is a bounded operator on a Banach space X then the following implications hold:*

(i) *If $T^* \in \Omega_-(X^*)$ then $T \in \Omega_+(X)$;*

(ii) *If $T^* \in \Omega_+(X^*)$ then $T \in \Omega_-(X)$.*

Proof (i) Suppose that $T \notin \Omega_+(X)$. We show that $T^* \notin \Omega_-(X^*)$. By hypothesis there exists an infinite-dimensional closed T-invariant subspace M of X such that $T|M$ is injective and$(T|M)^{-1}$ is continuous.

Let $M^\perp$ be the annihilator of M. Clearly, $M^\perp$ is closed and from the inclusion $T(M) \subseteq M$ it follows that $T^*(M^\perp) \subseteq M^\perp$. Moreover, the continuity of $(T|M)^{-1}$ implies that the operator $(T|M)^* : X^* \to M^*$ is surjective, see Theorem 1.6. Now, if $f \in X^*$ then the restriction $f|M \in M^*$ and there exists by surjectivity an element $g \in X^*$ such that $(T|M)^* g = f|M$, or, equivalently, $g(T|M) = f|M$. Hence, $f - gT \in M^\perp$, so

$$f - gT + T^* g \in M^\perp + T^*(X^*),$$

from which we conclude that $X^* = M^\perp + T^*(X^*)$. This last equality obviously implies that $Q_{M^\perp} T^*$ is surjective.

To show that $T^* \notin \Omega_-(X^*)$ it suffices to prove that codim $M^\perp = \infty$. Let (x_n) be a sequence linearly independent elements of M. Denote by $J : X \to X^{**}$ the canonical isomorphism. It is easily seen that

$$M^\perp \subseteq \bigcap_{n=1}^{\infty} \ker(Jx_n),$$

so the elements $Jx_1, Jx_2, \dots$ being linearly independent, $M^\perp$ is infinite-codimensional.

(ii) Assume that $T \notin \Omega_-(X)$. We show that $T^* \notin \Omega_+(X^*)$. By assumption there exists an infinite-codimensional closed T-invariant subspace $M \subseteq X$ such that $Q_M T$ is surjective. Therefore $X = M + T(X)$ and $T^*(M^\perp) \subseteq M^\perp$. We show that $T^*|M^\perp : M^\perp \to X^*$ is injective and has a bounded inverse.

Let us consider the canonical quotient map $T^M : X/M \to X/M$. From assumption T^M is surjective and hence its dual

$$(T^M)^* : (X/M)^* \to (X/M)^*$$

is injective and has continuous inverse. Define $J : (X/M)^* \to M^\perp$ by

$$J(f)(x) := f(\widetilde{x}) \quad \text{for all } f \in (X/M)^*, \quad x \in X.$$

Clearly J is an isomorphism of $(X/M)^*$ onto $M^\perp$. If $x \in X$ and $\phi \in M^\perp$ then

$$\begin{aligned}(T^*\phi)(x) &= \phi(Tx) = J^{-1}(\phi)(\widetilde{Tx}) = J^{-1}(\phi)(T^M\widetilde{x}) \\ &= (T^M)^*(J^{-1}(\phi))\widetilde{x} = (J(T^M)^*(J^{-1})(\phi))(x),\end{aligned}$$

from which we obtain $T^*|M^\perp = J(T^M)^*J^{-1}$. From this it follows that $T^*|M^\perp$ is injective and has bounded inverse. To show that $T^* \notin \Omega_+(X^*)$ we need to prove that $M^\perp$ has infinite dimension. From codim M = dim X/M we obtain that

$$\dim M^\perp = \dim J((X/M)^*) = \dim(X/M)^* = \infty,$$

so the proof is complete. ■

Theorem 3.19. *The class of all Riesz operator $\mathcal{R}(X)$ is contained in $\Omega_+(X) \cap \Omega_-(X)$.*

Proof We show first that $\mathcal{R}(X) \subseteq \Omega_+(X)$. Suppose that $T \notin \Omega_+(X)$. Then there exists a closed infinite-dimensional T-invariant subspace M of X such that the $T|M$ admits a bounded inverse. Trivially, $\alpha(T|M) = 0$ and $T(M)$ is closed, so $T|M \in \Phi_+(M)$.

Let us suppose that $T \in \mathcal{R}(X)$. By Theorem 2.71, part (iii), the restriction $T|M$ is still a Riesz operator so, by Remark 1.53 $T|M$ is not a Fredholm operator. Hence $\beta(T|M) = \infty$. From this it follows that ind $T|M = -\infty$ and the stability of the index implies that $\lambda I_M - T|M$ has index $-\infty$ in some annulus $0 < |\lambda| < \varepsilon$. This is impossible since $T|M$ is a Riesz operator. Therefore $\mathcal{R}(X) \subseteq \Omega_+(X)$.

We show now that $\mathcal{R}(X) \subseteq \Omega_-(X)$. We use an argument dual to that given in the first part of the proof.

Suppose that there is a Riesz operator T which does not belong to $\Omega_-(X)$. Then there exists a closed infinite-codimensional T-invariant subspace M of X such that the composition map $Q_MT : X \to X/M$ is surjective. Therefore the induced map T^M on X/M is onto and hence $T^M \in \Phi_-(X/M)$.

Now, by Theorem 2.73, T^M is a Riesz operator, and since X/M is infinite-dimensional then T^M is not a Fredholm operator. Consequently $\alpha(T^M) = \infty$, so $\operatorname{ind} T^M = \infty$ and the stability of the index yields that

$\lambda\widetilde{I}^M - T^M$ has index ∞ in some annulus $0 < |\lambda| < \varepsilon$. This contradicts the fact that T^M is a Riesz operator. Therefore $\mathcal{R}(X) \subseteq \Omega_-(X)$. ∎

Example 3.20. The inclusion $\mathcal{R}(X) \subseteq \Omega_+(X)$ and $\mathcal{R}(X) \subseteq \Omega_-(X)$ are generally proper. In fact, a well known result of Read [**95**] establishes that there exists a bounded operator T on ℓ^1 which does not admit a non-trivial closed T-invariant subspace. The spectrum $\sigma(T)$ of this operator is the whole closed unit disc and hence $T \notin R(\ell^1)$, whereas, obviously, $T \in \Omega_+(\ell^1)$. Moreover, since this operator T is not surjective we also have $T \in \Omega_-(\ell^1)$.

Another interesting example which shows that the inclusion $\mathcal{R}(X) \subseteq \Omega_+(X)$ is strict, is provided by a convolution operator on $L^1(\mathbb{T})$, where $\mathbb{T}$ is the circle group, see [**1**, Chapter 7].

We now characterize the Riesz operators among the classes $\Omega_+(X)$ and $\Omega_-(X)$.

Theorem 3.21. *For every operator $T \in L(X)$ on a Banach space X, the following statements are equivalent:*

(i) *T is a Riesz operator;*

(ii) *$T \in \Omega_+(X)$ and the spectrum $\sigma(T)$ is either a finite set or a sequence which converges to 0;*

(iii) *$T \in \Omega_-(X)$ and the spectrum $\sigma(T)$ is either a finite set or a sequence which converges to 0.*

Proof (i) $\Leftrightarrow$ (ii) The spectrum of a Riesz operator is finite or a sequence which clusters at 0 so the implication (i) $\Rightarrow$ (ii) is clear from Theorem 3.19.

Conversely, suppose that there exists an operator $T \in L(X)$ such that the condition (ii) is satisfied. Let λ be any spectral point different from 0 and denote by P_λ the spectral projection associated with the spectral set $\{\lambda\}$. By Theorem 2.68 to prove that T is Riesz it suffices to show that P_λ is a finite-dimensional operator. If we let $M_\lambda := P_\lambda(X)$, from the functional calculus we know that M_λ is a closed T-invariant subspace. Furthermore, we have $\sigma(T|M_\lambda) = \{\lambda\}$, so that $0 \notin \sigma(T|M_\lambda)$ and hence $T|M_\lambda$ is invertible. Since $T \in \Omega_+(X)$ this implies that $M_\lambda = P_\lambda(X)$ is finite-dimensional, as desired.

(i) $\Leftrightarrow$ (iii) The implication (i) $\Rightarrow$ (iii) is clear, again by Theorem 3.19. Conversely, suppose that there exists an operator $T \in L(X)$ such that the condition (iii) is satisfied. Let λ be any spectral point different from 0 and, as above, let P_λ be the spectral projection associated with the spectral set $\{\lambda\}$. Again, by Theorem 2.68, to prove that T is Riesz it suffices to show that P_λ is a finite-dimensional operator. If we let $M_\lambda := P_\lambda(X)$ and $N_\lambda := (I - P_\lambda)(X)$, then M_λ and N_λ are both T-invariant closed subspaces. Furthermore, from the functional calculus we have $X = M_\lambda \oplus N_\lambda$,

$$\sigma(T|M_\lambda) = \{\lambda\}, \quad \sigma(T|N_\lambda) = \sigma(T) \setminus \{\lambda\}.$$

We claim that $Q_{N_\lambda}T : X \to X/N_\lambda$ is surjective.

To see this observe first that for every $\widehat{x} \in X/N_\lambda$ there is $z \in X$ such that $Q_{N_\lambda}z = \widehat{x}$. From the decomposition $X = M_\lambda \oplus N_\lambda$ we know that there exist $u \in M_\lambda$ and $v \in N_\lambda$ such that $z = u + v$, and consequently,

$$\widehat{x} = Q_{N_\lambda}z = Q_{N_\lambda}u.$$

Since $0 \notin \{\lambda\} = \sigma(T|M_\lambda)$, the operator $T|M_\lambda : M_\lambda \to M_\lambda$ is bijective, so there exists $w \in M_\lambda$ such that

$$u = (T|M_\lambda)w = Tw.$$

From this it follows that

$$\widehat{x} = Q_{N_\lambda}u = (Q_{N_\lambda}T)w,$$

i.e, $Q_{N_\lambda}T$ is surjective. Since, by assumption, $T \in \Omega_-(X)$ it follows that codim $N_\lambda < \infty$, and hence dim $M_\lambda < \infty$. This show that P_λ is a finite-dimensional operator, so the proof is complete. ■

The class of Riesz operators may be also characterized as the set of operators for which the sum with a compact operator is an $\Omega_+(X)$ operator, as well as an $\Omega_-(X)$ operator.

Theorem 3.22. *If $T \in L(X)$ is an operator on a Banach space X, the following statements are equivalent:*

(i) *T is Riesz;*

(ii) *For every $K \in \mathcal{K}(X)$ we have $T + K \in \Omega_+(X)$;*

(iii) *For every $K \in \mathcal{K}(X)$ we have $T + K \in \Omega_-(X)$.*

Proof (i) $\Leftrightarrow$ (ii) If T is a Riesz operator also $T+K$ is a Riesz operator, by (iv) of Theorem 2.70, and hence $T + K \in \Omega_+(X)$ by Theorem 3.19.

Conversely, if T is not a Riesz operator there exists by Theorem 2.68 some $\lambda \neq 0$ such that $\lambda I - T \notin \Phi_+(X)$. Thus from part (i) of Theorem 1.57 we may find a compact operator $K \in L(X)$ such that

$\alpha(\lambda I - T - K) = \infty$. If $M := \ker(\lambda I - T - K)$ then M is infinite-dimensional and $(T + K)x = \lambda x$ for each $x \in M$. Therefore $T + K \notin \Omega_+(X)$.

(i) $\Leftrightarrow$ (iii) If T is a Riesz operator then $T + K$ is a Riesz operator, by (iv) of Theorem 2.70, and hence $T + K \in \Omega_-(X)$ by Theorem 3.19.

Conversely, if T is not a Riesz operator there exists by Theorem 2.68 some $\lambda \neq 0$ such that $\lambda I - T \notin \Phi_-(X)$. From part (ii) of Theorem 1.57 there exists a compact operator $K \in L(X)$ such that $\overline{\beta}(\lambda I - T - K) = \infty$. Let $M := \overline{(\lambda I - T - K)(X)}$. Then M is infinite-codimensional. Denote by $\widetilde{x}$ any residual class of X/M and define $(T + K)^M : X/M \to X/M$ as follows:

$$(T + K)^M \widetilde{x} := \widetilde{(T + K)x} \quad \text{where } x \in \widetilde{x}.$$

We show that $(T+K)^M$ is surjective. If $x \in X$ we have $(\lambda I - T - K)x \in M$, and consequently

$$(\lambda I \widetilde{- T} - K)\widetilde{x} = \widetilde{0}.$$

Therefore

$$\widetilde{x} = (T + K)^M(\lambda^{-1}\widetilde{x}) \quad \text{for all } x \in X.$$

Since the operator $(T + K)^M$ has the same range as the composition operator $Q_M(T + K)$, so, M being infinite-codimensional we conclude that $T + K \notin \Omega_-(X)$. ■

The class of all inessential operators $\mathcal{I}(X)$ is contained in each one of the two classes $\Omega_+(X)$ and $\Omega_-(X)$, since $\mathcal{I}(X) \subseteq \mathcal{R}(X)$. The following result due to Aiena [**3**] characterizes $\mathcal{I}(X)$ as the maximal ideal of $\Omega_+(X)$ and $\Omega_-(X)$-operators.

Theorem 3.23. *For every Banach space X, $\mathcal{I}(X)$ is the uniquely determined maximal ideal of $\Omega_+(X)$ operators. Each ideal of $\Omega_+(X)$ operators is contained in $\mathcal{I}(X)$. Analogously, $\mathcal{I}(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators. Each ideal of $\Omega_-(X)$ operators is contained in $\mathcal{I}(X)$.*

Proof Let G be any ideal of $\Omega_+(X)$ operators. Furthermore, let T be a fixed element of G and S any bounded operator on X. Then $ST \in G$, $\ker(I - ST)$ is a closed subspace invariant under ST, and the restriction of ST to $\ker(I - ST)$ coincides with the restriction of I to $\ker(I - ST)$. Therefore $ST|\ker(I - ST)$ has a bounded inverse. From the definition of $\Omega_+(X)$, $\ker(I - ST)$ is finite-dimensional, and hence, by Theorem 3.7, $T \in \mathcal{I}(X)$. Hence any ideal of $\Omega_+(X)$ operators is contained in $\mathcal{I}(X)$. On the other hand, $\mathcal{I}(X)$ is itself an ideal of $\Omega_+(X)$ operators, so $\mathcal{I}(X)$

is the uniquely determined maximal ideal of $\Omega_+(X)$ operators.

To show that $\mathcal{I}(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators, let us consider any ideal G of $\Omega_-(X)$ operators. If $T \in G$ and S is any bounded operator on X then $TS \in G$. Let $M := \overline{(I-TS)(X)}$. Then

$$[(TS(I-TS)](X) = [(I-TS)TS](X) \subseteq (I-TS)(X).$$

¿From this it follows that M is invariant under TS. Let us consider the induced quotient map $(TS)^M : X/M \to X/M$. The map $(TS)^M$ is surjective; in fact, if $x \in X$ then from $(I-TS)x \in M$ we obtain that $\widetilde{x} = \widetilde{(TS)x}$. From the definition of $\Omega_-(X)$ operators it follows that

$$\overline{\beta}(I-TS) = \text{codim } \overline{(I-TS)(X)} < \infty,$$

and thus by Theorem 3.8 $T \in \mathcal{I}(X)$. Since $\mathcal{I}(X)$ is itself an ideal of $\Omega_-(X)$ operators we conclude that $\mathcal{I}(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators. ∎

The following useful result characterizes the inessential operators acting between different Banach space by means of the two classes $\Omega_+(X)$ and $\Omega_-(X)$.

Theorem 3.24. *If $T \in L(X,Y)$, where X and Y are Banach spaces, then the following assertions are equivalent:*

(i) *$T \in \mathcal{I}(X,Y)$;*

(ii) *$ST \in \Omega_+(X)$ for all $S \in L(Y,X)$;*

(iii) *$TS \in \Omega_-(Y)$ for all $S \in L(Y,X)$.*

Proof (i) $\Leftrightarrow$ (ii) Let $T \in \mathcal{I}(X,Y)$ and $S \in L(Y,X)$. Then for each $\lambda \neq 0$ $\lambda I_X - ST \in \Phi(X)$, and hence $ST \in \mathcal{R}(X) \subseteq \Omega_+(X)$.

Conversely, let us suppose that $T \notin \mathcal{I}(X,Y)$. Then by Theorem 3.7 there exists an operator $S \in L(Y,X)$ such that $\alpha(I_X - ST) = \infty$. Let $M := \ker(I_X - ST)$. Clearly $ST|M = I_X|M$, and since M is infinite-dimensional we conclude that $ST \notin \Omega_+(X)$, and the proof of the equivalence (i) $\Leftrightarrow$ (ii) is complete.

(i) $\Leftrightarrow$ (iii) Let $T \in \mathcal{I}(X,Y)$ and $S \in L(Y,X)$. Then by Corollary 3.5 for each $\lambda \neq 0$ we have $\lambda I_Y - TS \in \Phi(Y)$, and hence $TS \in \mathcal{R}(X) \subseteq \Omega_-(Y)$.

Conversely, let us suppose that $T \notin \mathcal{I}(X,Y)$. Then by Theorem 3.8 there exists an operator $S \in L(Y,X)$ such that $\overline{\beta}(I_Y - TS) = \infty$. Let $M := \overline{(I-TS)(Y)}$. The induced quotient operator $(TS)^M : Y/M \to$

Y/M is surjective. In fact, for each $y \in Y$ we have $(I_Y - TS)y \in M$ and hence $(TS)^M \widetilde{y} = \widetilde{(TS)y}$. Since M is infinite-codimensional and $(TS)^M$ has the same range of $Q_M TS$ it then follows that $TS \notin \Omega_-(Y)$. ■

4. Φ-ideals

We want give in this section other examples of ideals of Riesz operators. The following definition is motivated by the classes of operators introduced in the previous sections.

Definition 3.25. *A two sided ideal $\mathcal{J}(X)$ of $L(X)$ is said to be a Φ-ideal if $\mathcal{F}(X) \subseteq \mathcal{J}(X)$ and $I - T \in \Phi(X)$ for all $T \in \mathcal{J}(X)$.*

Examples of Φ-ideals are $\mathcal{F}(X)$, $\mathcal{K}(X)$ and $\mathcal{I}(X)$. Evidently, every $T \in \mathcal{J}(X)$ is a Riesz operator, so by Theorem 3.23 and 3.19 the set $\mathcal{I}(X)$ is the maximal Φ-ideal of Riesz operators.

Other examples of Φ-ideals are provided by the perturbation classes of $\Phi_+(X)$ and $\Phi_-(X)$. To see this, let us consider the perturbation classes of $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$, i.e., if $\Phi_+(X,Y) \neq \varnothing$ then

$$P\Phi_+(X,Y) := \{T \in L(X,Y) : T + \Phi_+(X,Y) \subseteq \Phi_+(X,Y)\},$$

and, analogously, if $\Phi_-(X,Y) \neq \varnothing$ then

$$P\Phi_-(X,Y) := \{T \in L(X,Y) : T + \Phi_-(X,Y) \subseteq \Phi_-(X,Y)\}.$$

By Theorem 3.14, $P\Phi_+(X)$ and $P\Phi_-(X)$ are two-sided ideals and contain $\mathcal{F}(X)$. More precisely, by Theorem 1.56 we have

$$\mathcal{K}(X,Y) \subseteq P\Phi_+(X,Y) \quad \text{and} \quad \mathcal{K}(X,Y) \subseteq P\Phi_-(X,Y).$$

Theorem 3.26. *Suppose that X and Y are Banach spaces. Then we have*

(i) *If $\Phi_+(X,Y) \neq \varnothing$ then $P\Phi_+(X,Y) \subseteq \mathcal{I}(X,Y)$.*

(ii) *If $\Phi_-(X,Y) \neq \varnothing$ then $P\Phi_-(X,Y) \subseteq \mathcal{I}(X,Y)$.*

Moreover, $P\Phi_+(X)$ and $P\Phi_-(X)$ are Φ-ideals.

Proof (i) Assume that $T \in L(X,Y)$ and $T \notin \mathcal{I}(X,Y)$. By Theorem 3.7 there exists an operator $S \in L(Y,X)$ such that $I_X - ST \notin \Phi_+(X)$. On the other hand, for every operator $U \in \Phi_+(X,Y)$ we have $U(I_X - ST) = U - UST \notin \Phi_+(X,Y)$, otherwise we would have $I_X - ST \in \Phi_+(X,Y)$. But $U \in \Phi_+(X,Y)$ so $UST \notin P\Phi_+(X,Y)$, and hence by part (ii) of Theorem 3.14 we conclude that $T \notin P\Phi_+(X,Y)$.

To show the assertion (ii) assume that $T \in L(X,Y)$ and $T \notin \mathcal{I}(X,Y)$.

By Theorem 3.8 there exists $S \in L(Y, X)$ such that $I_Y - TS \notin \Phi_-(Y)$. For every operator $U \in \Phi_-(X, Y)$ it follows that

$$(I_Y - TS)U = U - TSU \notin \Phi_-(Y, X).$$

But $U \in \Phi_-(X, Y)$, so $TSU \notin P\Phi_-(X, Y)$, and hence $T \notin P\Phi_-(X, Y)$.

The last assertion is clear ∎

The inessential operators, as well the perturbation ideals are defined by means of the classes of Fredholm and semi-Fredholm operators. Finite-dimensional operators, as well as compact operators are defined by means of their action on certain subsets of the Banach space X. It is natural to ask if there exists an *intrinsic* characterization of these classes of operators, i.e. if an inessential operator T can be characterized by means the action of T on suitable subspaces. This is possible for special pairs of Banach spaces. To see this we shall introduce two other Φ-ideals. We recall that if M is a closed subspace of X by J_M we denote the canonical embedding, while by Q_M we denote the canonical quotient map from X onto X/M.

Definition 3.27. *Given two Banach spaces X and Y, an operator $T \in L(X, Y)$ is said to be* strictly singular *if no restriction TJ_M of T to an infinite-dimensional closed subspace M of X is an isomorphism.*

The operator $T \in L(X, Y)$ is said to be strictly cosingular *if there is no infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective.*

In the sequel we denote by $SS(X, Y)$ and $SC(X, Y)$ the classes of all strictly singular operators and strictly cosingular operators, respectively.

For the detailed study of the basic properties of the two classes $SS(X, Y)$ and $SC(X, Y)$ we refer the reader to Pietsch [**92**], Section 1.9 and Section 1.10 (these operators are also called *Kato operators* and *Pełczyński operators*, respectively).

We remark, however, that $SS(X, Y)$ and $SC(X, Y)$ are closed linear subspaces of $L(X, Y)$. Furthermore, if $T \in L(X, Y)$, $S \in SS(X, Y)$ (respectively, $S \in SC(X, Y)$ and $U \in L(Y, Z)$) then $UST \in SS(X, Z)$ (respectively, $UST \in SC(X, Z)$). Therefore $SS(X) := SS(X, X)$ and $SC(X) := SC(X, X)$ are closed ideals of $L(X)$.

Theorem 3.28. *$SS(X)$ and $SC(X)$ are Φ-ideals of $L(X)$.*

Proof Obviously ,

$$SS(X) \subseteq \Omega_+(X) \quad \text{and} \quad SC(X) \subseteq \Omega_-(X),$$

and hence by Theorem 3.23 $SS(X) \subseteq \mathcal{I}(X)$ and $SC(X) \subseteq \mathcal{I}(X)$. ∎

Lemma 3.29. *If X and Y are infinite-dimensional Banach spaces and $T \in L(X,Y)$ then the following implications hold:*

$$T \in \Phi_+(X,Y) \Rightarrow T \notin SS(X,Y), \quad T \in \Phi_-(X,Y) \Rightarrow T \notin SC(X,Y).$$

Proof If $T \in \Phi_+(X,Y)$ then $X = \ker T \oplus M$ for some closed subspace M of X, and $T_0 : M \to T(X)$, defined by $T_0 x := Tx$ for all $x \in M$, is an isomorphism between infinite-dimensional closed subspaces. Hence $T \notin SS(X,Y)$.

Analogously, if $T \in \Phi_-(X,Y)$ then $Y = T(X) \oplus N$ for some infinite-codimensional closed subspace N. Evidently

$$Y/N = Q_N(Y) = Q_N T(X),$$

thus $T \notin SC(X,Y)$. ∎

Theorem 3.30. *If X and Y are Banach spaces then*

$$\mathcal{K}(X,Y) \subseteq SS(X,Y) \cap SC(X,Y).$$

Proof Suppose that $T \notin SS(X,Y)$ and $T \in \mathcal{K}(X,Y)$. Then there exists a closed infinite-dimensional subspace M of X such that TJ_M admits a bounded inverse. Since TJ_M is compact and has closed range $T(M)$, it follows that $T(M)$ is finite-dimensional by Theorem 1.38, and hence also M is finite-dimensional, a contradiction. This shows the inclusion $\mathcal{K}(X,Y) \subseteq SS(X,Y)$.

Analogously, suppose that $T \notin SC(X,Y)$ and $T \in \mathcal{K}(X,Y)$. Then there exists a closed infinite-codimensional subspace N of Y such that $Q_N T$ is onto Y/N. But $T \in \mathcal{K}(X,Y)$ so $Q_N T$ is compact, and since its range is closed then, again by Theorem 1.38, Y/N is finite-dimensional. Thus N is finite-codimensional, a contradiction. Hence $\mathcal{K}(X,Y) \subseteq SC(X,Y)$. ∎

Example 3.31. A strictly singular operator can have a non-separable range, see Goldberg and Thorp [**57**, p. 335], in contrast to the well known result that every compact operator has separable range. Hence the inclusion $\mathcal{K}(X,Y) \subseteq SS(X,Y)$ in general is proper.

Theorem 3.32. *If X and Y are Banach spaces and $\Phi_+(X,Y) \neq \emptyset$ then $SS(X,Y) \subseteq P\Phi_+(X,Y)$. Analogously, if $\Phi_-(X,Y) \neq \emptyset$ then $SC(X,Y) \subseteq P\Phi_-(X,Y)$*

Proof Assume that $\Phi_+(X,Y) \neq \varnothing$ and suppose that $T \in SS(X,Y)$ and $S \in \Phi_+(X,Y)$. We need to show that $T+S \in \Phi_+(X,Y)$. Suppose that $T+S \notin \Phi_+(X,Y)$. By part (i) of Theorem 1.57 we know that there exists a compact operator $K \in \mathcal{K}(X,Y)$ such that $\alpha(T+S-K)=\infty$. Set $M := \ker(T+S-K)$. Then M is a closed infinite-dimensional subspace and

$$TJ_M = (K-S)J_M.$$

If $U := K-S$ then $U \in \Phi_+(X,Y)$ because $S \in \Phi_+(X,Y)$ and K is compact. Hence $TJ_M = UJ_M \in \Phi_+(M,Y)$, and from Lemma 3.29 we obtain that $TJ_M \notin SS(M,Y)$. From this it follows that $T \notin SS(X,Y)$, a contradiction. Therefore the inclusion $\mathcal{SS}(X,Y) \subseteq P\Phi_+(X,Y)$ is proved.

To show the second assertion, let assume that $\Phi_-(X,Y) \neq \varnothing$ and suppose that $T \in SC(X,Y)$ and $S \in \Phi_-(X,Y)$. We need to show that $T+S \in \Phi_-(X,Y)$. Suppose that $T+S \notin \Phi_-(X,Y)$. By Theorem 1.57, part (ii), then there exists a compact operator $K \in \mathcal{K}(X,Y)$ such that $M := \overline{(T+S-K)(X)}$ is a closed infinite-codimensional subspace of Y. Denote by $\widetilde{x}$ the residual class $x+M$ in X/M. From $(T+S-K)x \in M$ we deduce that

$$Q_M(T+S-K)x = \widetilde{0} \quad \text{for all } x \in X.$$

Therefore $Q_M T = Q_M(K-S)$. Clearly, if $U := K-S$ then $U \in \Phi_-(X,Y)$, and hence $Q_M U \in \Phi_-(X,Y/M)$. By Lemma 3.29 we then infer that $Q_M T = Q_M U \notin SC(X,Y/M)$ and this implies that $T \notin SC(X,Y)$. ■

The next example shows that the classes $SS(X,Y)$ and $SC(X,Y)$ may coincide with $L(X,Y)$.

Example 3.33. Let $p,q \in \mathbb{N}$ be such that $1 < p,q < \infty$ and $p \neq q$. Then $SS(\ell^p,\ell^q) = SC(\ell^p,\ell^q) = L(\ell^p,\ell^q)$, see [**1**, Chapter 7] for details.

We show next that if someone of the two Banach spaces possesses many complemented subspaces, in the sense of the following definitions, then the inclusions of Theorem 3.26 actually are equalities.

Definition 3.34. *A Banach space X is said to be* subprojective *if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace which is complemented in X.*

A Banach space X is said to be superprojective *if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace which is complemented in X.*

In the next examples we see that most of the classical Banach spaces are subprojective or superprojective.

Example 3.35. We list some examples of subprojective and superprojective Banach spaces.

(a) Evidently every Hilbert space is both subprojective and superprojective. The spaces ℓ^p with $1 < p < \infty$, and c_0 are subprojective, see Whitley [**106**, Theorem 3.2]. Moreover, all the spaces ℓ^p with $1 < p < \infty$ are superprojective.

(b) The spaces $L^p[0,1]$, with $2 \leq p < \infty$, are subprojective and $L^p[0,1]$, with $1 < p < 2$ are not subprojective, see Whitley [**106**, Theorem 3.4]. Moreover, all spaces $L^p[0,1]$ with $1 < p \leq 2$ are superprojective and are not superprojective for $2 < p < \infty$.

(c) The spaces $L^1[0,1]$ and $C[0,1]$ are neither subprojective nor superprojective, see Whitley [**106**, Corollary 3.6].

The next theorem provides an internal characterization of $\mathcal{I}(X,Y)$, $P\Phi_+(X,Y)$ and $P\Phi_-(X,Y)$ for most of the classical Banach spaces.

Theorem 3.36. *Let X, Y be Banach spaces. The following statements hold:*

(i)*) If Y is subprojective then $SS(X,Y) = P\Phi_+(X,Y) = \mathcal{I}(X,Y)$;*

(ii) *If X is superprojective then $SC(X,Y) = P\Phi_-(X,Y) = \mathcal{I}(X,Y)$.*

Proof (i) Let $T \in L(X,Y)$, $T \notin SS(X,Y)$. Then there exists an infinite-dimensional closed subspace M of X such that the restriction TJ_M is bounded below. Since Y is subprojective we can assume that the subspace $T(M)$ is complemented in Y, namely $Y = T(M) \oplus N$. Then $X = M \oplus T^{-1}(N)$ and defining $A \in L(Y,X)$ equal to T^{-1} in $T(M)$ and equal to 0 in N it easily follows that AT is a projection of X onto M. Therefore $\ker(I_X - AT) = (AT)(X) = M$ is infinite-dimensional, so we have $I_X - AT \notin \Phi(X)$ and consequently $T \notin \mathcal{I}(X,Y)$.

(ii) Let $T \in L(X,Y)$, $T \notin SC(X,Y)$. Then there exists a closed infinite-codimensional subspace N of Y such that $Q_N T$ is surjective. Note that $T^{-1}(N)$ is infinite-codimensional because Y/N is isomorphic to $X/T^{-1}(N)$. So since X is subprojective we can assume that $T^{-1}(N)$ is complemented in X. Clearly $\ker T \subset T^{-1}(N)$, and hence if $X = M \oplus T^{-1}(N)$ then $T(M) \cap N = \{0\}$. Moreover, $Y = T(X) + N$ implies $Y = T(M) + N$, from which it follows that $T(M)$ is closed, see Theorem 1.7. Therefore the topological direct sum $Y = T(M) \oplus N$ is satisfied.

Now, if we define $A \in L(Y, X)$ as in part (i) we then easily obtain that $T \notin \mathcal{I}(X, Y)$. ■

Theorem 3.37. *If X and Y are Banach spaces and $T^* \in SS(Y^*, X^*)$, then $T \in SC(X, Y)$. Analogously, if $T^* \in SC(Y^*, X^*)$ then $T \in SS(X, Y)$.*

Proof The proof is analogous to that of Theorem 3.18. ■

We complement the result of Theorem 3.37 by mentioning a result established by Pełczyński [**91**], which shows that T^* is strictly singular whenever $T \in L(X, Y)$ is strictly cosingular and weakly compact. It is an open problem if T^* is strictly cosingular whenever $T \in L(X, Y)$ is strictly singular and weakly compact, or, equivalently, if T is strictly singular and Y is reflexive.

Theorem 3.38. *Let X and Y be Banach spaces and $T \in L(X, Y)$. Then the following statements hold:*

(i) *If $T^* \in SS(Y^*, X^*)$ and Y is subprojective then $T \in SS(X, Y)$;*

(ii) *If $T \in SS(Y, X)$, where X is reflexive and X^* is subprojective, then $T^* \in SS(X^*, Y^*)$.*

Proof (i) Suppose that $T \notin SS(X, Y)$. Then there is an infinite-dimensional closed subspace M of X such that $T|M$ has a bounded inverse. Since Y is subprojective we can find a closed infinite-dimensional subspace N of $T(M)$ and a bounded projection P of Y onto N. Note that $N = T(W)$, where W is a closed infinite-dimensional subspace of M. Since P maps Y onto N its dual P^* is an injective continuous map of N^* onto an infinite-dimensional subspace Z of Y^*. We prove that $T^*|Z$ has a bounded inverse and is therefore not strictly singular. For any $f \in Z$ we have

$$\|T^*f\| = \sup_{0 \neq x \in X} \frac{\|(T^*f)(x)\|}{\|x\|} \geq \sup_{0 \neq x \in W} \frac{\|f(Tx)\|}{\|x\|}.$$

Since $T|M$ has a bounded inverse there is $K > 0$ such that $\|Tx\| \geq K\|x\|$ for all $x \in W \subseteq M$. Moreover, since $f \in P^*(N^*)$ there is $g \in N^*$ such that $P^*g = f$. Combining this information, we deduce that

$$\|T^*f\| \geq \sup_{0 \neq y \in N} \frac{K\|(P^*g)y\|}{\|y\|} = \sup_{0 \neq y \in N} \frac{K\|g(Py)\|}{\|y\|} = K\|g\|.$$

Finally, from $\|P^*\|\|g\| \geq \|P^*g\| = \|f\|$ we obtain $\|T^*f\| \geq (K/\|P^*\|)\|f\|$. This shows that T^* has a bounded inverse on the subspace Z of Y^*, so that $T^* \notin SS(Y^*, X^*)$.

(ii) Note that $T^{**} = J_Y T {J_X}^{-1}$, where J_X and J_Y are the canonical embeddings of, respectively, X onto X^{**} and Y into Y^{**}. Then if T is strictly singular so is T^{**}, and by part (i) also T^* is strictly singular. ■

Corollary 3.39. *If X is a Hilbert space and if $T \in SS(X,Y)$ then also T^* is strictly singular. If Y is a Hilbert space and if $T^* \in SS(Y^*, X^*)$, then T is strictly singular.* ■

5. Further and recent developments

It is well-known that every separable Banach space is isomorphic to a subspace of $C[0,1]$. Consequently, the hypothesis in the following theorem implies that $\Phi_+(X,Y) \neq \varnothing$, so we can consider the perturbation class $P\Phi_+(X,Y)$.

Theorem 3.40. *Suppose that X is separable and Y contains a complemented subspace isomorphic to $C[0,1]$. Then $P\Phi_+(X,Y) = SS(X,Y)$.*

Again, every separable Banach space is isomorphic to a quotient of ℓ^1. Therefore, the hypothesis in the following theorem implies that $\Phi_-(X,Y) \neq \varnothing$.

Theorem 3.41. *Suppose that X contains a complemented subspace isomorphic to ℓ^1 and Y is separable. Then*

$$P\Phi_-(X;Y) = SC(X,Y).$$

Both Theorem 3.40 and Theorem 3.41 have been proved in [**20**], see also Chapter 7 of [**1**].

It has been for a long time an open problem whether or not the equalities $\mathcal{SS}(X,Y) = P\Phi_+(X,Y)$ and $\mathcal{SC}(X,Y) = P\Phi_-(X,Y)$ hold for all Banach spaces X, Y. Next we present a.recent result of González [**59**] which gives an example of a Banach space for which both equalities do not hold. Before we need some definitions and some preliminary work.

Definition 3.42. *A Banach space X is said to be* decomposable *if it contains a pair of infinite-dimensional closed subspaces M and N, such that $X = M \oplus N$. Otherwise X is said to be* indecomposable. *A Banach space X is said to be* hereditarily indecomposable *if every closed subspace* *of X is indecomposable, i.e., there no exist infinite-dimensional closed subspaces M and N for which $M \cap N = \{0\}$ and $M + N$ is closed. Dually, a Banach space X is said to be* quotient hereditarily indecomposable *if every quotient of X is indecomposable.*

By duality it is easy to prove that if X^* is hereditarily indecomposable (respectively, quotient hereditarily indecomposable) then X is quotient hereditarily indecomposable (respectively, hereditarily indecomposable), but the converse implication are not valid. Finite-dimensional Banach spaces are trivial examples of indecomposable spaces. Note that the existence of infinite-dimensional indecomposable Banach spaces has been a long standing open problem and has been positively solved by Gowers and Maurey [**61**] and [**62**], who construct an example of a reflexive, separable, hereditarily indecomposable Banach space X_{GM}. Moreover, the dual X_{GM}^* is quotient hereditarily indecomposable. Later, Ferenczi [**56**] proved that X_{GM} is quotient hereditarily indecomposable. Moreover, Gowers and Maurey constructed in [**62**] a whole family of other indecomposable Banach spaces, amongst them we shall find some Banach spaces useful in order to construct our counterexamples.

Roughly speaking one can say that indecomposable Banach spaces are Banach spaces with small spaces of operators. In fact, we have ([**61**])

Theorem 3.43. *If a complex Banach space X is hereditarily indecomposable then $L(X) = \{\mathbb{C}I_X\} \oplus SS(X)$, while if X is quotient hereditarily indecomposable, then $L(X) = \{\mathbb{C}I_X\} \oplus SC(X)$.*

The Gowers Maurey construction is rather technical and requires some techniques of analysis combined with involved combinatorial arguments. We do not describe this space, but in the sequel we shall point out some of the properties of these spaces, from the point of view of Fredholm theory, needed for the construction of the mentioned counterexample.

Hereditarily indecomposable and quotient hereditarily indecomposable Banach spaces may be characterized by Fredholm theory. The following result was proved by Weis [**104**] when it was not known the existence of hereditarily indecomposable and quotient hereditarily indecomposable Banach spaces.

Theorem 3.44. *Let X be a Banach space. Then we have:*

(i) *$L(X,Y) = \Phi_+(X,Y) \cup SS(X,Y)$ for all Banach spaces Y if and only if X is hereditarily indecomposable.*

(ii) $L(Y,X) = \Phi_-(Y,X) \cup SC(Y,X)$ *for all Banach spaces* Y *if and only if* Y *is quotient hereditarily indecomposable.*

A nice consequence of Theorem 3.44 is that for a hereditarily indecomposable Banach space X with scalar field $\mathbb{K}$ then $L(X)/SS(X)$ is a division algebra, and dually, if X is quotient hereditarily indecomposable then $L(X)/SC(X)$ is a division algebra [**55**]. It is well-known that if the scalar field $\mathbb{K} = \mathbb{C}$ then $\mathbb{C}$ is the unique division algebra, while if $\mathbb{K} = \mathbb{R}$ there are three division algebras, $\mathbb{R}$, $\mathbb{C}$ and the quaternion algebra, see the book by Bonsall and Duncan [**39**, §14].

The next result is some way appears surprising. In Theorem 3.36 we have seen that if X or Y possesses many complemented subspaces then $SS(X,Y) = P\Phi_+(X,Y) = \mathcal{I}(X,Y)$ and $SC(X,Y) = P\Phi_-(X,Y) = \mathcal{I}(X,Y)$. The following result, due to Aiena and González [**18**] and [**19**], shows that the equalities $P\Phi_+(X,Y) = SS(X,Y)$ and $P\Phi_-(X,Y) = SC(X,Y)$ are still valid if X possesses very few complemented subspaces.

Corollary 3.45. *Suppose that* X *is hereditarily indecomposable,* Y *is any Banach space and* $\Phi_+(X,Y) \neq \varnothing$. *Then* $P\Phi_+(X,Y) = SS(X,Y)$. *Analogously, if* $\Phi_-(X,Y) \neq \varnothing$ *then* $P\Phi_-(X,Y) = SC(X,Y)$.

We can now establish the mentioned González's counter-example.

Theorem 3.46. *Let* X *be a reflexive hereditarily indecomposable Banach space and* Y *be a closed subspace of* X *such that* $\dim Y = \operatorname{codim} Y = \infty$. *If* $Z := X \times Y$ *then* $P\Phi_+(Z) \neq SS(Z)$ *and* $P\Phi_-(Z) \neq SC(Z)$.

In [**62**] Gowers and Maurey also produced an example of indecomposable complex Banach space which admits a Schauder basis. This space allows us to construct a counterexample for which another old question in Fredholm theory was negatively solved. To discuss this question, we need before to go back to the problem of finding an intrinsic characterization of inessential operators.

The following classes of operators has been introduced by Tarafdar [**100**] and [**101**]. Most of the results that we give in the sequel have been established by Aiena and González [**18**], [**19**].

Definition 3.47. *An operator* $T \in L(X,Y)$ *is said to be* improjective *if there exists no infinite-dimensional closed subspace* M *of* X

such that the restriction TJ_M is an isomorphism and $T(M)$ is a complemented subspace of Y. The set of all improjective operators from X into Y will be denoted by $\text{Imp}(X,Y)$ *and we set* $\text{Imp}(X) := \text{Imp}(X,X)$.

Trivially, the identity on an infinite-dimensional Banach space X and any isomorphism between infinite-dimensional Banach spaces are examples of operators which are not improjective. Moreover, the restriction $T|M$ of an improjective operator $T \in \text{Imp}(X,Y)$ to a closed infinite-dimensional subspace M of X is also improjective. The class $\text{Imp}(X,Y)$ admits the following dual characterization in terms of quotient maps.

Theorem 3.48. *An operator $T \in L(X,Y)$ is improjective if and only if there is no infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of X.*

Inessential operators are improjective:

Theorem 3.49. *For every pair of Banach spaces X and Y we have $\mathcal{I}(X,Y) \subseteq \text{Imp}(X,Y)$.*

It should be noted that if $A \in L(Y,Z)$, $T \in \text{Imp}(X,Y)$ and $B \in L(W,X)$, then $TB \in \text{Imp}(W,Y)$ and $AT \in \text{Imp}(X,Z)$. Moreover, $\text{Imp}(X,Y)$ is a closed subset of $L(X,Y)$. We now investigate the cases where $\text{Imp}(X,Y) = \mathcal{I}(X,Y)$. Observe first that there is a symmetry:

Theorem 3.50. *Let X, Y be Banach spaces. Then $\mathcal{I}(X,Y) = \text{Imp}(X,Y)$ if and only if $\mathcal{I}(Y,X) = \text{Imp}(Y,X)$.*

The next result gives an intrinsic characterization of inessential operators whenever X or Y is subprojective or superprojective.

Theorem 3.51. *Assume that one of the spaces X, Y is subprojective or superprojective. Then we have* $\text{Imp}(X,Y) = \mathcal{I}(X,Y)$.

In particular, Theorem 3.51 applies to the classical Banach spaces listed in Example 3.35.

Corollary 3.52. *Suppose that X and Y are Banach spaces. Then the following assertions hold:*

(i) *If Y is subprojective then* $\text{Imp}(X,Y) = SS(X,Y)$;

(ii) *If X is superprojective then* $\text{Imp}(X,Y) = SC(X,Y)$.

Proof Combine Theorem 3.51 with Theorem 3.36. ∎

The following examples show that we cannot change the order of the spaces X, Y in Theorem 3.52. To be precise, in part (i) if X is subprojective then $\mathrm{Imp}(X, Y) = SS(X, Y)$ is not true in general, and analogously in part (ii) if Y is superprojective, then $\mathrm{Imp}(X, Y) = SC(X, Y)$ is in general not true.

Example 3.53. (a) The space ℓ^2 is subprojective, so by Theorem 3.51 and taking into account what was established in Example 3.10, we obtain

$$L(\ell^2, \ell^\infty) = \mathrm{Imp}(\ell^2, \ell^\infty) = \mathcal{I}(\ell^2, \ell^\infty);$$

On the other hand, we also have $L(\ell^2, \ell^\infty) \neq SS(\ell^2, \ell^\infty)$ because ℓ^∞ contains a closed subspace isomorphic to ℓ^2, see Beauzamy [**29**, Theorem IV.II.2] .

(b) The space ℓ^2 is superprojective and, as already seen, every operator $T \in L(\ell^1, \ell^2)$ is strictly singular. Therefore we have

$$L(\ell^1, \ell^2) = \mathrm{Imp}(\ell^1, \ell^2) = \mathcal{I}(\ell^1, \ell^2).$$

However, $L(\ell^1, \ell^2) \neq SC(\ell^1, \ell^2)$, because ℓ^1 has a quotient isomorphic to ℓ^2, see Beauzamy [**29**, Theorem IV.II.1].

In the following result due to Aiena and González [**19**] the indecomposability of complex Banach spaces has been characterized by means of improjective operators and Atkinson operators acting on it:

Theorem 3.54. *For a Banach space Y, the following statements are equivalent:*

(a) Y is indecomposable;

(b) $L(Y, Z) = \Phi_l(Y, Z) \cup Imp(Y, Z)$ for every Banach space Z;

(c) $L(X, Y) = \Phi_r(X, Y) \cup Imp(X, Y)$ for every Banach space X;

(d) $L(Y) = \Phi(Y) \cup Imp(Y)$.

By Corollary 3.52 if X is superprojective or Y is subprojective then $\mathrm{Imp}(X, Y) = \mathcal{I}(X, Y)$. Curiously, also in the case of Banach spaces which contain very few complemented subspaces all the improjective operators are inessential ([**19**, Corollary 3.4 and Proposition 3.5]). In fact we have:

Theorem 3.55. *Assume that the Banach space X is either hereditarily indecomposable or quotient indecomposable. Then $L(X) = \Phi(X) \cup \mathcal{I}(X)$ and $Imp(X, Y) = \mathcal{I}(X, Y)$, for every Y.*

It is a natural question whether the equality $\mathcal{I}(X,Y) = \text{Imp}(X,Y)$ holds for all Banach spaces X and Y. Another natural question was whether $\text{Imp}(X,Y)$ is a linear subspace of $L(X,Y)$. Clearly, the answer to the last question is positive if $\mathcal{I}(X,Y) = \text{Imp}(X,Y)$, as in the cases considered above. In 1972, Tarafdar [**100, 101**] gave some partial answers, including some of the previous results. But since that time, these relevant questions remained unsolved for many years. In [**19**] Aiena and Gonzalez produced an example of a Banach space that shows that the answers to the previous questions are negative. To see this counter-example observe first that Theorem 3.54 and Theorem 3.55 suggest a breakthrough in the possibility of finding a solution of the Tarafdar question: an example of improjective operator which is not inessential could exist in the case of an indecomposable Banach spaces X which is neither hereditarily indecomposable nor quotient indecomposable.
We first note that such a space does exist:

Theorem 3.56. [**62**] *There exists an indecomposable Banach space X_s which is neither hereditarily indecomposable nor quotient indecomposable. This space has a Schauder basis and the associated right shift S is an isometry on X_s.*

The Banach space X_s provides the counter-example required:

Theorem 3.57. *([**19**]) For the Gowers-Maurey space X_s the following assertions hold:*

a) $\mathcal{I}(X_s) \neq \text{Imp}(X_s)$.

b) *There exists $T \in \text{Imp}(X_s)$ which is not Riesz.*

c) $\text{Imp}(X_s)$ *is not a subspace of $L(X_s)$.*

Proof It is well-known that for the isometric right shift operator S, $\lambda I - S$ is invertible for every $|\lambda| > 1$ and hence is a Fredholm operator with ind $(\lambda I - S) = 0$. On the other hand we also have $\lambda I - S \in \Phi(X_s)$, with ind $(\lambda I - S) = -1$, for every $|\lambda| < 1$. From the continuity of the index of Fredholm operators then $\lambda I - S$ cannot be a Fredholm operator for $|\lambda| = 1$. Now, by Theorem 3.54 $\lambda I - S$ is improjective for every $|\lambda| = 1$. Note that $T := \lambda I - S$ is not a Riesz operator; in particular it is not inessential. Moreover, since X_s is infinite dimensional, we have $I \notin \text{Imp}(X_s)$, and from the equality

$$(I - S) + (I + S) = 2I$$

we find two improjective operators whose sum is not improjective. ∎

Note that if for a Banach space X we have $\mathrm{Imp}(X) \neq \mathcal{I}(X)$ then $\mathrm{Imp}(X, Y) \neq \mathcal{I}(X, Y)$ for all Banach spaces Y. In fact, we have ([**18**]):

Theorem 3.58. *For a Banach space X the following statements are equivalent:*

(i) *$Imp(X) = \mathcal{I}(X)$;*

(ii) *$Imp(X, Y) = \mathcal{I}(X, Y)$ for all Banach spaces Y;*

(iii) *$Imp(X) \subseteq \Omega_+(X)$;*

(iv) *$Imp(X) \subseteq \Omega_-(X)$;*

(v) *$Imp(X) \subseteq \mathcal{R}(X)$.*

One may ask if every $T \in \mathrm{Imp}(X)$, X a complex Banach space, whose spectrum is either finite or a sequence which clusters at 0, is an inessential operator. The answer to this question is still negative, since the counterexample given in Theorem 3.57 allows us to construct an operator which disproves this conjecture, see ([**19**]) for details.

Theorem 3.59. *There exists a complex Banach space X and a quasi-nilpotent improjective operator $T \in L(X)$ such that $T \notin \mathcal{I}(X)$.*

An important field in which Fredholm theory finds a natural application is that of the incomparability of Banach spaces. There are several notions of incomparability; for an excellent survey we refer to González and Martinón [**60**]. Roughly speaking, two Banach spaces X and Y are incomparable if there is no isomorphism between *certain* infinite-dimensional subspaces.

Definition 3.60. *Two Banach spaces X and Y are said to be* projection incomparable*, or also* totally dissimilar*, if no infinite-dimensional complemented subspace of X is isomorphic to a complemented subspace of Y.*

The next result shows that the notion of incomparabilty defined above may be given in terms of improjective operators.

Theorem 3.61. *Two Banach spaces X and Y are projection incomparable precisely when $L(X, Y) = \mathrm{Imp}(X, Y)$.*

Another notion of incomparability is the following one introduced by González [**58**].

Definition 3.62. *Two Banach spaces X and Y are said to be* essentially incomparable *if $L(X, Y) = \mathcal{I}(X, Y)$.*

¿From the inclusion $\mathcal{I}(X,Y) \subseteq \mathrm{Imp}(X,Y)$ we immediately obtain:

X, Y essentially incomparable $\Rightarrow$ X, Y projection incomparable.

Moreover, since the existence of $T \in \Phi(X,Y)$ implies that $\ker T$ has an infinite-dimensional complement M isomorphic to $T(X)$ we also have:

$$X, Y \text{ projection incomparable } \Rightarrow \Phi(X,Y) = \varnothing.$$

The last implication, in general, cannot be reversed. In fact, if $X = L^p[0,1]$ and $Y = L^q[0,1]$, with $1 < p < q < \infty$, then $\Phi(X,Y) = \varnothing$, whereas $\mathcal{I}(X,Y) \neq L(X,Y)$, since both $L^p[0,1]$ and $L^q[0,1]$ have a complemented subspace M isomorphic to ℓ^2, M the subspace spanned by the Rademacher functions, see Lindenstrauss and Tzafriri [**73**].

Of course, all the examples in which $L(X,Y) = \mathcal{I}(X,Y)$, provide pairs of Banach spaces which are essentially incomparable. Analogously, all the examples of Banach spaces for which $L(X,Y) = \mathrm{Imp}(X,Y)$ provide examples of projection incomparable Banach spaces. The next result is an obvious consequence of Theorem 3.51.

Theorem 3.63. *Suppose that X or Y is a subprojective Banach space. Then X and Y are projection incomparable precisely when X and Y are essentially incomparable. Analogously, if X or Y is superprojective then X and Y are projection incomparable if and only if X and Y are essentially incomparable.* ∎

It should be noted that the two kinds of incomparability are not the same. In fact, the example given in Theorem 3.57 also allows us to show that the inequality $\mathcal{I}(Z,Y) \neq L(Z,Y)$ does not imply that Z has an infinite-dimensional complemented subspace isomorphic to a complemented subspace of Y. Moreover, we see also that $\mathrm{Imp}(Z,Y)$ being a subspace of $L(Z,Y)$ does not imply $\mathcal{I}(Z,Y) = \mathrm{Imp}(Z,Y)$.

Theorem 3.64. *There exist a pair of Banach spaces Z, Y for which we have*

$$\mathcal{I}(Z,Y) \neq \mathrm{Imp}(Z,Y) = L(Z,Y).$$

The construction of such a pair Z, Y may be found in Aiena and González [**19**].

CHAPTER 4

Spectral theory

The spectrum of a bounded linear operator T on a Banach space X can be splitted into subsets in many different ways, depending on the purpose of the inquiry. In this chapter we look more closely to some parts of the spectrum of a bounded operator on a Banach space from the viewpoint of Fredholm theory. In particular, we study some parts of the spectrum, as the Weyl spectra and the Browder. We also introduce some special classes of operators having nice spectral properties. These operators include those for which the so-called Browder's theorem and Weyl's theorem hold. In particular, we see that the localized single-valued extension property studied in Chapter 2 is a precious tool in order to produce a general framework for the classes of operators satisfying the theorems above mentioned. Browder's theorem and Weyl's theorem admit some variants, as a-Browder's theorem, a-Weyl's theorem and property (w) that we study in last sections of these notes.

1. Weyl and Browder spectra

Given a bounded operator $T \in L(X)$ the *Fredholm spectrum*, in literature also called *essential spectrum*, is defined by

$$\sigma_{\mathrm{f}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

By Corollary 1.52 we know that $\sigma_{\mathrm{f}}(T)$ coincides with the spectrum of $\widehat{T} := T + \mathcal{K}(X)$ in the Calkin algebra $L(X)/\mathcal{K}(X)$, and hence, by Remark 2.4, if X is an infinite-dimensionsal complex Banach space then $\sigma_{\mathrm{f}}(T) \neq \varnothing$.

Recall that the *approximate point spectrum* is defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below,}\}$$

while *surjectivity spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}.$$

By Theorem 1.6 we have $\sigma_{\rm a}(T) = \sigma_{\rm s}(T^*)$ and $\sigma_{\rm s}(T) = \sigma_{\rm a}(T^*)$. The classes of Weyl operators generate the following spectra:

1) the *Weyl spectrum* defined by

$$\sigma_{\rm w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *upper semi-Weyl spectrum* defined by

$$\sigma_{\rm uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *lower semi-Weyl spectrum* defined by

$$\sigma_{\rm lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

Note that $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm a}(T)$ and $\sigma_{\rm lw}(T) \subseteq \sigma_{\rm s}(T)$.

The results of the following theorem easily follows from duality.

Theorem 4.1. *Let $T \in L(X)$. Then we have:*

(i) $\sigma_{\rm w}(T) = \sigma_{\rm w}(T^*)$,

(ii) $\sigma_{\rm uw}(T) = \sigma_{\rm lw}(T^*)$ *and* $\sigma_{\rm lw}(T) = \sigma_{\rm uw}(T^*)$. *Moreover,*

$$\sigma_{\rm w}(T) = \sigma_{\rm uw}(T) \cup \sigma_{\rm lw}(T).$$

¿From Theorem 2.57 and Theorem 2.59 we also have

Theorem 4.2. *Let $T \in L(X)$. Then we have*

$$\sigma_{\rm uw}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma_{\rm a}(T+K), \quad \sigma_{\rm lw}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma_{\rm s}(T+K), \tag{33}$$

and

$$\sigma_{\rm w}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K). \tag{34}$$

It should be noted that, always by Theorem 2.57 and Theorem 2.59, that the ideal $\mathcal{K}(X)$ may be replaced by $\mathcal{F}(X)$ (actually, by every Φ-ideal !).

Of course, also the classes of Browder operators generate spectra. The *Browder spectrum* defined by

$$\sigma_{\rm b}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* defined by

$$\sigma_{\rm ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\},$$

and the *lower semi-Browder spectrum* defined by

$$\sigma_{\rm lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_-(X)\}.$$

Also the following results follows by duality.

Theorem 4.3. *Let $T \in L(X)$. Then we have:*
(i) $\sigma_{\rm b}(T) = \sigma_{\rm b}(T^*)$.
(ii) $\sigma_{\rm ub}(T) = \sigma_{\rm lb}(T^*)$ *and* $\sigma_{\rm lb}(T) = \sigma_{\rm ub}(T^*)$. *Moreover,*

$$\sigma_{\rm b}(T) = \sigma_{\rm ub}(T) \cup \sigma_{\rm lb}(T).$$

¿From Theorem 2.62, Theorem 2.63 and Theorem 2.64 we also have:

Theorem 4.4. *Let $T \in L(X)$. Then we have*
(35)
$$\sigma_{\rm ub}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma_{\rm a}(T+K), \quad \sigma_{\rm lb}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma_{\rm s}(T+K),$$

and

(36)
$$\sigma_{\rm b}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma(T+K).$$

Evidently, the following inclusions hold:

$$\sigma_{\rm f}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T),$$

so, if X is an infinite-dimensional complex Banach space, $\sigma_{\rm w}(T)$ and $\sigma_{\rm b}(T)$ are non-empty compact subsets of $\mathbb{C}$. Again,

$$\sigma_{\rm uw}(T) \subseteq \sigma_{\rm ub}(T) \quad \text{and} \quad \sigma_{\rm lw}(T) \subseteq \sigma_{\rm lb}(T),$$

and from Remark 1.53 it then easily follows that also $\sigma_{\rm uw}(T)$ and $\sigma_{\rm lw}(T)$, as well as $\sigma_{\rm ub}(T)$ and $\sigma_{\rm lb}(T)$, are non-empty compact subsets of $\mathbb{C}$. The precise relationships between Weyl spectra and Browder spectra are established in the following theorem:

Theorem 4.5. *For a bounded operator $T \in L(X)$ the following statements hold:*
(i) $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{\rm a}(T)$.
(ii) $\sigma_{\rm lb}(T) = \sigma_{\rm lw}(T) \cup \operatorname{acc} \sigma_{\rm s}(T)$.
(iii) $\sigma_{\rm b}(T) = \sigma_{\rm w}(T) \cup \operatorname{acc} \sigma(T)$.

Proof (i) If $\lambda \notin \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{\rm a}(T)$ then $\lambda I - T \in \Phi_+(X)$ and $\sigma_{\rm a}(T)$ does not cluster at λ. By Theorem 2.51 then T has SVEP at λ, so by Theorem 2.45 $p(\lambda I - T) < \infty$ and hence $\lambda \notin \sigma_{\rm ub}(T)$. This shows the inclusion $\sigma_{\rm ub}(T) \subseteq \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{\rm a}(T)$.

Conversely, suppose that $\lambda \in \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{\rm a}(T)$. If $\lambda \in \sigma_{\rm uw}(T)$ then $\lambda \in \sigma_{\rm ub}(T)$, since $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm ub}(T)$. If $\lambda \in \operatorname{acc} \sigma_{\rm a}(T)$ then $\lambda \in \sigma_{\rm uw}(T)$ or $\lambda \notin \sigma_{\rm uw}(T)$. In the first case $\lambda \in \sigma_{\rm ub}(T)$. In the second case

$\lambda I - T \in \Phi_+(X)$, so by Theorem 2.51 T does not have the SVEP at λ, and hence $p(\lambda I - T) = \infty$ by Theorem 2.45. From this we conclude that $\lambda \in \sigma_{\rm ub}(T)$. Therefore the equality (i) is proved.

The proof of the equality (ii) is similar. The equality (iii) follows combining (i) with (ii) and taking into account the equality $\sigma(T) = \sigma_{\rm a}(T) \cup \sigma_{\rm s}(T)$. ∎

In the particular case that T or T^* has SVEP we can say more:

Theorem 4.6. *Suppose that $T \in L(X)$.*

(i) *If T has SVEP then $\sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma_{\rm lb}(T)$.*

(ii) *If T^* has SVEP then $\sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma_{\rm ub}(T)$.*

(iii) *If either T or T^* has the SVEP. Then*

$$\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T) \quad \textit{and} \quad \sigma_{\rm lw}(T) = \sigma_{\rm lb}(T).$$

Proof (i) We show the inclusion $\sigma_{\rm b}(T) \subseteq \sigma_{\rm w}(T)$. If $\lambda \notin \sigma_{\rm w}(T)$ then $\lambda I - T \in W(X)$ and the SVEP ensures, by Theorem 2.45 that $p(\lambda I - T) < \infty$. Since $\lambda I - T \in W(X)$ it the follows, by Theorem 1.21, that $q(\lambda I - T) < \infty$, thus $\lambda \notin \sigma_{\rm b}(T)$. The opposite inclusion holds for every $T \in L(X)$, so $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$. To show the equality $\sigma_{\rm b}(T) = \sigma_{\rm lb}(T)$ we need only to prove that $\sigma_{\rm b}(T) \subseteq \sigma_{\rm lb}(T)$. If $\lambda \notin \sigma_{\rm lb}(T)$ then $\lambda I - T \in \Phi_-(X)$ and $q(\lambda I - T) < \infty$. The SVEP implies by Theorem 2.45 that $p(\lambda I - T) < \infty$, hence by Theorem 1.21 $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, thus $\lambda \notin \sigma_{\rm b}(T)$.

(ii) The equalities may be shown arguing as in part (i), just use Theorem 2.46 instead of Theorem 2.45.

(iii) Suppose first that T has the SVEP. By part (ii) of Theorem 4.5, to show that $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$, it suffices to prove that $\mathrm{acc}\, \sigma_{\rm a}(T) \subseteq \sigma_{\rm uw}(T)$. Suppose that $\lambda \notin \sigma_{\rm uw}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and the SVEP at λ ensures that $\sigma_{\rm a}(T)$ does not cluster at λ, by Theorem 2.51. Hence $\lambda \notin \mathrm{acc}\, \sigma_{\rm a}(T)$.

To prove the equality $\sigma_{\rm lb}(T) = \sigma_{\rm lw}(T)$ it suffices to show that $\sigma_{\rm lb}(T) \subseteq \sigma_{\rm lw}(T)$. Suppose that $\lambda \notin \sigma_{\rm lw}(T)$. Then $\lambda I - T \in \Phi_-(X)$ with $\beta(\lambda I - T) \le \alpha(\lambda I - T)$. Again, the SVEP at λ gives $p(\lambda I - T) < \infty$, and hence by part (i) of Theorem 1.21 $\alpha(\lambda I - T) = \beta(\lambda I - T)$. At this point the finiteness of $p(\lambda I - T)$ implies by part (iv) of Theorem 1.21 that also $q(\lambda I - T)$ is finite, so $\lambda \notin \sigma_{\rm lb}(T)$. Therefore $\sigma_{\rm lb}(T) \subseteq \sigma_{\rm lw}(T)$, and the proof of the second equality is complete in the case that T has the SVEP.

Suppose now that T^* has SVEP. Then by the first part $\sigma_{\rm ub}(T^*) =$

$\sigma_{\rm uw}(T^*)$ and $\sigma_{\rm lb}(T^*) = \sigma_{\rm lw}(T^*)$. By duality it follows that $\sigma_{\rm lb}(T) = \sigma_{\rm lw}(T)$ and $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$.

The equality $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$ follows from $\sigma_{\rm b}(T) = \sigma_{\rm ub}(T) \cup \sigma_{\rm lb}(T)$ and $\sigma_{\rm w}(T) = \sigma_{\rm uw}(T) \cup \sigma_{\rm lw}(T)$. ■

By Theorem 2.77 and Theorem 2.81 Browder spectra and Weyl spectra are invariant under Riesz commuting perturbations.

Theorem 4.7. *Suppose that $T \in L(X)$ and R a Riesz operator commuting with T. Then we have*

(i) $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(T+R)$ *and* $\sigma_{\rm lw}(T) = \sigma_{\rm lw}(T+R)$.

(ii) $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(T+R)$ *and* $\sigma_{\rm lb}(T) = \sigma_{\rm lb}(T+R)$.

(iii) $\sigma_{\rm b}(T) = \sigma_{\rm b}(T+R)$ *and* $\sigma_{\rm w}(T) = \sigma_{\rm w}(T+R)$.

It has some sense to ask whenever the approximate point spectrum is stable under Riesz commuting perturbations. The answer is negative, also for commuting finite-dimensional perturbations. However, we have the following result:

Theorem 4.8. *Suppose that $T \in L(X)$ and $K \in L(X)$ a finite-dimensional operator commuting with T. Then we have*

(i) $acc\, \sigma_{\rm a}(T) = acc\, \sigma_{\rm a}(T+K)$.

(ii) $acc\, \sigma_{\rm s}(T) = acc\, \sigma_{\rm s}(T+K)$.

Proof We show first $\operatorname{acc} \sigma_{\rm a}(T+K) = \sigma_{\rm a}(T)$.Suppose first that T is injective. We show that $K(X) \subseteq T(X)$. Let $\{y_1, \dots y_n\}$ be a basis of $K(X)$. Then $\{Ty_1, \dots Ty_n\}$ is a linearly independent, since T is injective. Moreover, $Ty_j \in K(X)$ since if $y_j = Ky_j$ for some $x_j \in X$ then $Ty_j = TKy_j = KTy_j \in K(X)$. Since $K(X)$ is the subspace generated by $\{Ty_1, \dots Ty_n\}$ it then follows that $K(X) \subseteq T(X)$. Suppose now that $\mu \notin \operatorname{acc} \sigma_{\rm a}(T)$ and choose $\varepsilon > 0$ such that for all $0 < |\lambda - \mu| < \varepsilon$ we have $\lambda I - T$ is bounded below. Also, there exists a bounded operator $T_1 : (\lambda I - T)(X) \to X$ such that $(\lambda I - T)T_1$ is the restriction of I on $(\lambda I - T)(X)$ while $T_1(\lambda I - T) = I_X$. Since $K(X)$ is a finite-dimensional subspace of the Banach space $(\lambda I - T)(X)$ we may find a closed subspace M such that $K(X) \oplus M = (\lambda I - T)(X)$. Suppose that $\lambda \in \sigma_{\rm a}(T+K)$. Then there exists a sequence $(x_n) \subset X$ such that $\|x_n\| = 1$ and $(\lambda I - (T+K))x_n \to 0$. We can assume that $Kx_n \to x \in K(X)$. Now,

$$0 = \lim T_1(\lambda I - (T+K))x_n = \lim(x_n + T_1Kx_n),$$

and since $T_k x_n \to T_1 x$ we obtain $x_n \to -T_1 x$. Since $\|x_n\| = 1$ it follows that $x \neq 0$. Clearly,

$$x = \lim K x_n = -K T_1 x \in K(X).$$

Also, $(\lambda I - T)x = -(\lambda I - T)KT_1 x = -Kx$ and $(\lambda I - (T + K))x = 0$. Hence if $\lambda \in \sigma_{\mathrm{a}}(T + K)$ then λ is an eigenvalue of $T + K$. It is known that eigenvectors corresponding to distinct eigenvalues of $T + K$ are linearly independent and this contradicts the finite dimension of $K(X)$. Therefore $\sigma_{\mathrm{a}}(T+K)$ may contain only finitely many points λ such that $0 < |\lambda - \mu| < \varepsilon$. Therefore $\mu \notin \operatorname{acc} \sigma_{\mathrm{a}}(T + K)$. The opposite inclusion easily follows by symmetry.

(ii) Follows by duality. ∎

Theorem 4.9. *Suppose that $T \in L(X)$ and Q a quasi-nilpotent operator commuting with T. Then $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + Q)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{s}}(T + Q)$.*

Proof The inclusion $\sigma_{\mathrm{a}}(T+S) \subseteq \sigma_{\mathrm{a}}(T)+\sigma_{\mathrm{a}}(S)$ holds for all commuting operators $T, S \in L(X)$, see [**76**, p.256]. Therefore, $\sigma_{\mathrm{a}}(T+Q) \subseteq \sigma_{\mathrm{a}}(T) + \{0\} = \sigma_{\mathrm{a}}(T)$. The opposite inclusion is obtained by symmetry: $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + Q - Q) \subseteq \sigma_{\mathrm{a}}(T + Q)$. The equality $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{s}}(T + Q)$ follows by duality. ∎

2. Regularities

Let $\mathcal{H}(\sigma(T))$ be the set of all complex-valued functions which are locally analytic on an open set containing $\sigma(T)$. In Chapter 2 for every $f \in \mathcal{H}(\sigma(T))$ we have defined by the functional calculus the operator

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1} d\lambda.$$

It is well-known that the spectral mapping theorem holds for T, i.e. ([**68**, Theorem 48.2]) :

Theorem 4.10. *If $T \in L(X)$, X a Banach space, then $\sigma(f(T)) = f(\sigma(T))$.*

It is natural to ask whether the spectral theorem holds for some of the spectra previously defined. To answer to this question we introduce, in the more general context of Banach algebra the concept of regularity. Let $\mathcal{A}$ be an unital Banach algebra with unit u, and let denote by $\operatorname{inv} \mathcal{A}$ the set of all invertible elements.

Definition 4.11. *A non-empty subset* $\mathcal{R}$ *of* $\mathcal{A}$ *is said to be a* regularity *if the following conditions are satisfied:*

(i) $a \in \mathcal{R} \Leftrightarrow a^n \in \mathcal{R}$ *for all* $n \in \mathbb{N}$.

(ii) *If* a, b, c, d *are mutually commuting elements of* $\mathcal{A}$ *and* $ac+bd = u$ *then*

$$ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R} \text{ and } b \in \mathcal{R}.$$

It is easily seen that if $\mathcal{R}$ is a regularity then $u \in \mathcal{R}$ and $\operatorname{inv} \mathcal{A} \subseteq \mathcal{R}$. Moreover,if $a, b \in \mathcal{A}$, $ab = ba$, and $a \in \operatorname{inv} \mathcal{A}$ then

$$ab \in \mathcal{R} \Leftrightarrow b \in \mathcal{R}. \tag{37}$$

In fact $a\, a^{-1} + b\, 0 = u$, so the property (ii) above applies.

It is easy to verify the following criterion.

Theorem 4.12. *Let* $\mathcal{R} \neq \varnothing$ *be a subset of* $\mathcal{A}$. *Suppose that for all commuting elements* $a, b \in \mathcal{A}$ *we have*

$$ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R} \text{ and } b \in \mathcal{R}. \tag{38}$$

Then $\mathcal{R}$ *is a regularity.*

Denote by

$$\sigma_{\mathcal{R}}(a) := \{\lambda \in \mathbb{C} : \lambda u - a \notin \mathcal{R}\},$$

the *spectrum corresponding to the regularity* $\mathcal{R}$. Obviously, $\operatorname{inv} \mathcal{A}$ is a regularity by Theorem 4.12, and the corresponding spectrum is the ordinary spectrum. Note that $\sigma_{\mathcal{R}}(a)$ may be empty. For instance if $\mathcal{A} := L(X)$ and $\mathcal{R} = L(X)$. The proof of the following result is immediate.

Theorem 4.13. *The intersection* $\mathcal{R}$ *of a family* $(\mathcal{R}_\alpha)_\alpha$ *of regularities is again a regularity. Moreover,*

$$\sigma_{\mathcal{R}}(a) = \bigcup_\alpha \sigma_{\mathcal{R}_\alpha}(a), \quad a \in \mathcal{A}.$$

The union $\mathcal{R}$ *of a directed system of regularities* $(\mathcal{R}_\alpha)_\alpha$ *is again a regularity. Moreover,*

$$\sigma_{\mathcal{R}}(a) = \bigcap_\alpha \sigma_{\mathcal{R}_\alpha}(a), \quad a \in \mathcal{A}.$$

Now we state the spectral mapping theorem for regularities:

Theorem 4.14. *Suppose that $\mathcal{R}$ is a regularity in a Banach algebra $\mathcal{A}$ with unit u. Then $\sigma_{\mathcal{R}}(f(a)) = f(\sigma_{\mathcal{R}}(a))$ for every $a \in \mathcal{A}$ and every $f \in \mathcal{H}(\sigma(a))$ which is non-constant on each component of its domain of definition.*

Proof It is sufficient to prove that

$$\mu \notin \sigma_{\mathcal{R}}(f(a)) \Leftrightarrow \mu \notin f(\sigma_{\mathcal{R}}(a)). \tag{39}$$

Since $f(\lambda)-\mu$ has only a finite number of zeros $\lambda_1, \dots, \lambda_n$ in the compact set $\sigma(a)$ then we can write

$$f(\lambda) - \mu = (\lambda - \lambda_1)^{\nu_1} \cdots (\lambda - \lambda_n)^{\nu_n} \cdot g(\lambda),$$

where g is an analytic function defined on an open set containing $\sigma(a)$ and $g(\lambda) \neq 0$ for $\lambda \in \sigma(a)$. Then $f(a) - \mu u = (a - \lambda_1 u)^{\nu_1} \cdots (a - \lambda_n u)^{\nu_n} \cdot g(a)$, with $g(a)$ invertible by the spectral mapping theorem for the ordinary spectrum. Therefore, (39) is equivalent to

$$f(a) - \mu u \in \mathcal{R} \Leftrightarrow a - \lambda_k u \in \mathcal{R} \quad \text{for all } k = 1, \dots n. \tag{40}$$

But $g(a)$ is invertible, so, by (37) and the definition of a regularity, (40) is equivalent to saying

$$(a - \lambda_1 u)^{\nu_1} \cdots (a - \lambda_n u)^{\nu_n} \in \mathcal{R} \Leftrightarrow (a - \lambda_k)^{\nu_k} \in \mathcal{R} \quad \text{for all } k = 1, \dots n \tag{41}$$

Since for all relatively prime polynomials p, q there exist polynomials p_1, q_1 such that $pp_1 + qq_1 = 1$ we have $p(a)p_1(a) + q(a)q_1(a) = u$ and applying property (ii) of Definition 4.11 we then obtain, by induction, the equivalence (41). ∎

In the assumptions of Theorem 4.14 the condition that f is non constant on each component cannot dropped. In fact, the spectral mapping theorem for constant functions cannot be true if $\sigma_{\mathcal{R}}(a) = \varnothing$ for some $a \in \mathcal{A}$. and $0 \notin \mathcal{R}$.

Let us consider a regularity $\mathcal{R}(X) \subseteq L(X)$ and let X_1, X_2 be a pair of closed subspaces of X for which $X = X_1 \oplus X_2$. Define

$$\mathcal{R}_1 := \{T_1 \in L(X_1) : T_1 \oplus I_{X_2} \in \mathcal{R}\}$$

and

$$\mathcal{R}_2 := \{T_2 \in L(X_2) : I_{X_1} \oplus T_2 \in \mathcal{R}\},$$

It is easy to see that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are regularity in $L(X_1)$ and $L(X_2)$, respectively. Moreover, $T_1 \oplus T_2 \in \mathcal{R}(X) \Leftrightarrow T_1 \in \mathcal{R}_1(X_1)$ and $T_2 \in \mathcal{R}_2(X_2)$. We also have $\sigma_{\mathcal{R}}(T_1 \oplus T_2) = \sigma_{\mathcal{R}_1}(T_1) \cup \sigma_{\mathcal{R}_2}(T_2)$. The proof of the following result, together other properties of regularities, may be found in [**84**, Chapter 6]

Theorem 4.15. *Let $\mathcal{R}$ be a regularity in $L(X)$, and suppose that for all closed subspaces X_1 and X_2, $X = X_1 \oplus X_2$, such that the regularity $\mathcal{R}_1 \neq L(X_1)$ and $\sigma_{\mathcal{R}_1}(T_1) \neq \varnothing$ for all $T_1 \in L(X_1)$. Then*

$$\sigma_{\mathcal{R}}(f(T)) = f(\sigma_{\mathcal{R}}(T))$$

for every $T \in L(X)$ and every $f \in \mathcal{H}(\sigma(T))$.

In many situations a regularity decomposes as required in Theorem 4.15. For instance if $\mathcal{R} := \{T \in L(X) : T \text{ is onto}\}$ and $X = X_1 \oplus X_2$ then $\mathcal{R}_i = \{T_i \in L(X_i) : T_i \text{ is onto}\}$, $i = 1, 2$, and $T_1 \oplus T_2$ is onto if and only if T_1, T_2 are onto. Thus the spectral mapping theorem for all $f \in \mathcal{H}(\sigma(T))$ is reduced, by Theorem 4.15 to the question on the non-emptiness of the spectrum.

Let us consider the following sets:

(1) $\mathcal{R}_1 := \{T \in L(X) : T \text{ is bounded below}\}$. In this case $\sigma_{\mathcal{R}_1}(T) = \sigma_{\mathrm{a}}(T)$.

(2) $\mathcal{R}_2 := \{T \in L(X) : T \text{ is onto}\}$. In this case $\sigma_{\mathcal{R}_2}(T) = \sigma_{\mathrm{s}}(T)$.

(3) $\mathcal{R}_3 := \Phi_+(X)$. The corresponding spectrum is the *upper semi-Fredholm spectrum* $\sigma_{\mathrm{usf}}(T)$, known in literature also as *the essential approximate point spectrum.*

(4) $\mathcal{R}_4 := \Phi_-(X)$. The corresponding spectrum is the *lower semi-Fredholm spectrum* $\sigma_{\mathrm{lsf}}(T)$, known in literature also as *the essential surjective spectrum.*

(5) $\mathcal{R}_5 := B_+(X)$. The corresponding spectrum is $\sigma_{ub}(T)$.

(6) $\mathcal{R}_6 := B_-(X)$. The corresponding spectrum is $\sigma_{lb}(T)$.

(7) $\mathcal{R}_7 := \{T \in L(X) : T \text{ is semi-regular}\}$. In this case $\sigma_{\mathcal{R}_7}(T)$ is called *the Kato spectrum.*

(8) $\mathcal{R}_8 := \{T \in L(X) : T \text{ is essentially semi-regular}\}$. In this case $\sigma_{\mathcal{R}_8}(T)$ is the *essentially semi-regular spectrum.*

(9) $\mathcal{R}_9 := \{T \in L(X) : T \text{ is Saphar}\}$. In this case $\sigma_{\mathrm{sa}}(T) := \sigma_{\mathcal{R}_9}(T)$ is called the *Saphar spectrum.*

All the sets $\mathcal{R}_i$, $i = 1, 2, \dots, 9$ are regularities. The spectral mapping theorem holds for all these spectra:

Theorem 4.16. *If $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$ then $\sigma_{\mathcal{R}_i}(f(T)) = f(\sigma_{\mathcal{R}_i}(T))$, for all $i = 1, 2, \ldots, 9$.*

Proof All the regularities $\mathcal{R}_i$ satisfy the conditions of Theorem 4.15, see Chapter III of [**84**]. The reader may also find a proof of the spectral mapping theorem for these spectra, not involving the concept of regularity, in [**1**]. ■

For an arbitrary operator $T \in L(X)$ on a Banach space X let

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\} .$$

¿From the identity theorem for analytic functions it readily follows that $\Xi(T)$ is open and consequently is contained in the interior of the spectrum $\sigma(T)$. Clearly $\Xi(T)$ is empty precisely when T has the SVEP. The next result proved by Aiena, Miller and Neumann [**12**] shows that the spectral mapping theorem holds for $\Xi(T)$, see also [**1**, Chapter 2]

Theorem 4.17. *Let $T \in L(X)$, X a Banach space. Let $f : \mathcal{U} \to \mathbb{C}$ be an analytic function on the open neighborhood $\mathcal{U}$ of $\sigma(T)$. Suppose that f is non-constant on each of the connected components of $\mathcal{U}$. Then $f(T)$ has the SVEP at $\lambda \in \mathbb{C}$ if and only if T has the SVEP at every point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. Moreover, $f(\Xi(T)) = \Xi((f(T))$.*

An important consequence of Theorem 4.17 is given by the following result ([**1**, Chap 2]):

Theorem 4.18. *Let $T \in L(X)$, X a Banach space, and $f : \mathcal{U} \to \mathbb{C}$ an analytic function on the open neighborhood $\mathcal{U}$ of $\sigma(T)$. If T has the SVEP then $f(T)$ has the SVEP. If f is non-constant on each of the connected components of $\mathcal{U}$, then T has the SVEP if and only if $f(T)$ has the SVEP.*

The spectral mapping theorem does not hold for Weyl spectra. We only have

$$\sigma_{\mathrm{uw}}(f(T)) \subseteq f(\sigma_{uw}(T)) \quad \sigma_{\mathrm{lw}}(f(T)) \subseteq f(\sigma_{lw}(T)),$$

and

$$\sigma_{\mathrm{w}}(f(T)) \subseteq f(\sigma_{w}(T))$$

for all $f \in \mathcal{H}(\sigma(T))$, see Theorem 3.63, Theorem 3.67 of [**1**]. This inclusions may be strict, see Example 3.64 of [**1**].

However we have the following result.

Theorem 4.19. *Suppose that for $T \in L(X)$ either T or T^* has SVEP. Then the spectral theorem holds for $\sigma_{\mathrm{uw}}(T)$, $\sigma_{\mathrm{lw}}(T)$ and $\sigma_{\mathrm{w}}(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof If T has SVEP then $f(T)$ has SVEP and hence, by Theorem 4.6,

$$\sigma_{\mathrm{uw}}(f(T)) = \sigma_{\mathrm{ub}}(f(T)) = f(\sigma_{\mathrm{ub}}(T)) = f(\sigma_{\mathrm{uw}}(T)).$$

If T^* has SVEP then $f(T^*) = f(T)^*$ has SVEP, hence by Theorem 4.6

$$\sigma_{\mathrm{lw}}(f(T)^*) = \sigma_{\mathrm{lb}}(f(T)^*)) = f(\sigma_{\mathrm{lb}}(T^*) = f(\sigma_{\mathrm{lw}}(T^*)).$$

By duality then $\sigma_{\mathrm{uw}}(f(T)) = f(\sigma_{\mathrm{uw}}(T))$, as desired.
The spectral mapping theorem for $\sigma_{\mathrm{lw}}(T)$ and $\sigma_{\mathrm{w}}(T)$ may be proved in a similar way. ∎

3. Browder's theorem

For a bounded operator $T \in L(X)$ let us define

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \in \mathcal{B}(X)\},$$

the set of all *Riesz points* in $\sigma(T)$, and let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty.\}$$

Finally, let us consider the following set:

$$\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T).$$

Evidently, if $\lambda \in \Delta(T)$ then $\lambda I - T \in W(X)$ and since $\lambda \in \sigma(T)$ it follows that $\alpha(\lambda I - T) = \beta(\lambda I - T) > 0$, so we can write

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W(X),\, 0 < \alpha(\lambda I - T)\}.$$

Lemma 4.20. *For every $T \in L(X)$ we have $p_{00}(T) \subseteq \pi_{00}(T) \cap \Delta(T)$.*

Proof If $\lambda \in p_{00}(T)$ then $\lambda I - T \in B(X)$ and $p(\lambda I - T) = q(\lambda I - T) < \infty$, so λ is isolated in $\sigma(T)$. Furthermore, since $\lambda I - T \in W(X)$, we have $0 < \alpha(\lambda I - T)$, otherwise by Theorem 1.21, we would have $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, hence $\lambda \notin \sigma(T)$, a contradiction. ∎

The following concept has been introduced in 1997 by Harte and W. Y. Lee [**67**].

Definition 4.21. *A bounded operator T is said to satisfy* Browder's theorem *if*

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T),$$

or equivalently, by Theorem 4.5, if

$$acc\,\sigma(T) \subseteq \sigma_{\mathrm{w}}(T). \tag{42}$$

Browder's theorem is satisfied by several classes of operators:

Theorem 4.22. *Suppose that T or T^* has SVEP. Then Browder's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof It is immediate from Theorem 4.6, since $f(T)$ or $f(T)^*$ has SVEP by Theorem 4.18. ■

In general, Browder's theorem and the spectral mapping theorem are independent. In [**67**, Example 6] is given an example of an operator T for which the spectral mapping theorem holds for $\sigma_{\mathrm{w}}(T)$ but Browder's theorem fails for T. Another example [**67**, Example 7] shows that there exist operators for which Browder's theorem holds while the spectral mapping theorem for the Weyl spectrum fails.

The following result shows that Browder's theorem is equivalent to the localized SVEP at the points of the complement in $\mathbb{C}$ of $\sigma_{\mathrm{w}}(T)$.

Theorem 4.23. *For a bounded operator $T \in L(X)$ following statements are equivalent:*

(i) $p_{00}(T) = \Delta(T)$;

(ii) *T satisfies Browder's theorem;*

(iii) *T^* satisfies Browder's theorem;*

(iv) *T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$;*

(v) *T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$.*

Proof (i) ⇒ (ii) Suppose that $p_{00}(T) = \Delta(T)$. Let $\lambda \notin \sigma_{\mathrm{w}}(T)$ be arbitrary. We show that $\lambda \notin \sigma_{\mathrm{b}}(T)$. If $\lambda \notin \text{acc}\,\sigma(T)$ then by Theorem 4.5 we have $\lambda \notin \sigma_{\mathrm{b}}(T)$. Consider the other case $\lambda \in \text{acc}\,\sigma(T)$. Since $\sigma(T)$ is closed then $\lambda \in \sigma(T)$ and since $\lambda I - T \in W(X)$ it must be $0 < \alpha(\lambda I - T) = \beta(\lambda I - T)$. Therefore $\lambda \in \Delta(T) = p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, thus $\lambda \notin \sigma_{\mathrm{b}}(T)$. Hence $\sigma_{\mathrm{b}}(T) \subseteq \sigma_{\mathrm{w}}(T)$ and since the reverse inclusion is satisfied by every operator we then conclude that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$, i.e. T satisfies Browder's theorem.

(ii) ⇔ (iii) Obvious, since $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T^*)$ and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$.

(ii) ⇒ (iv) Assume that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$. If $\lambda \notin \sigma_{\mathrm{w}}(T)$ then $\lambda I - T \in B(X)$ so $p(\lambda I - T) < \infty$ and hence T has SVEP at λ.

(iv) ⇒ (v) Suppose that T has SVEP at every point $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{w}}(T)$. For every $\lambda \notin \sigma_{\mathrm{w}}(T)$ then $\lambda I - T \in W(X)$, and the SVEP at λ implies that $p(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ it then follows by Theorem 1.21 that $q(\lambda I - T) < \infty$, and consequently, T^* has SVEP at λ.

(v) $\Rightarrow$ (i) Suppose that $\lambda \in \Delta(T)$. We have $\lambda I - T \in W(X)$ and hence $\text{ind}(\lambda I - T) = 0$. By Theorem 2.46 the SVEP of T^* at λ implies that $q(\lambda I - T) < \infty$ and hence, again by Theorem 1.21, also $p(\lambda I - T)$ is finite. Therefore $\lambda \in \sigma(T) \setminus \sigma_{\text{b}}(T) = p_{00}(T)$. This shows that $\Delta(T) \subseteq p_{00}(T)$ and by Lemma 4.20 we then conclude that equality $p_{00}(T) = \Delta(T)$ holds. ■

The following example shows that SVEP for T or T^* is a not necessary condition for Browder's theorem.

Example 4.24. Let $T := L \oplus L^* \oplus Q$, where L is the unilateral left shift on $\ell^2(\mathbb{N})$, defined by

$$L(x_1, x_2, \dots) := (x_2, x_3, \cdots), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and Q is any quasi-nilpotent operator. Note that L is surjective but not injective, so by Corollary 2.25 L does not have SVEP, so also T and T^* do not have SVEP, see Theorem 2.9 of [**1**]. On the other hand, we have $\sigma_{\text{b}}(T) = \sigma_{\text{w}}(T) = \mathbf{D}$, where $\mathbf{D}$ is the closed unit disc in $\mathbb{C}$, thus Browder' theorem holds for T.

Let us write iso K for the set of all isolated points of $K \subseteq \mathbb{C}$. A very clear spectral picture of operators for which Browder's theorem holds is given by the following theorem:

Theorem 4.25. *For an operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies Browder's theorem;*

(ii) *Every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$;*

(iii) *$\Delta(T) \subseteq \partial\sigma(T)$, $\partial\sigma(T)$ the topological boundary of $\sigma(T)$;*

(iv) *$\text{int}\, \Delta(T) = \emptyset$;*

(v) *$\sigma(T) = \sigma_{\text{w}}(T) \cup \text{iso}\, \sigma(T)$.*

Proof (i) $\Rightarrow$ (ii) If T satisfies Browder's theorem then $\Delta(T) = p_{00}(T)$, and in particular every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Clear, since $\text{int}\, \partial\sigma(T) = \emptyset$.

(iv) $\Rightarrow$ (v) Suppose that $\text{int}\, \Delta(T) = \emptyset$. Let $\lambda_0 \in \Delta(T) = \sigma(T) \setminus \sigma_{\text{w}}(T)$. We show first that $\lambda_0 \in \partial\sigma(T)$. Suppose that $\lambda_0 \notin \partial\sigma(T)$. Then there exists an open disc centered at λ_0 contained in the spectrum. Since $\lambda_0 I - T \in W(X)$ by the classical punctured neighborhood theorem there exists another open disc $\mathbb{D}$ centered at λ_0 such that $\lambda I - T \in W(X)$ for all $\lambda \in \mathbb{D}$. Therefore $\lambda_0 \in \operatorname{int} \Delta(T)$, which is impossible. This argument shows that $\sigma(T) = \sigma_{\mathrm{w}}(T) \cup \partial\sigma(T)$.

Now, if $\lambda \in \partial\sigma(T)$ and $\lambda \notin \sigma_{\mathrm{w}}(T)$ then $\lambda I - T \in W(X)$ and, since both T and T^* have SVEP at every point of $\partial\sigma(T) = \partial\sigma(T^*)$, by Theorem 2.45 and Theorem 2.46 we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Therefore λ is an isolated point of $\sigma(T)$ and this entails that $\sigma(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{iso} \sigma(T)$.

(v) $\Rightarrow$ (i) Suppose that $\sigma(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{iso} \sigma(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Then $\lambda \in \operatorname{iso} \sigma(T)$. Since T and T^* have SVEP at every isolated point of $\sigma(T)$ and $\lambda I - T \in W(X)$ it then follows that $p(\lambda I - T) = q(\lambda I - T) < \infty$, so $\lambda \notin \sigma_{\mathrm{b}}(T)$. Therefore $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$. ■

Let $\Theta(M, N)$ be the gap between two closed subspaces M and N of a Banach space X, defined in the first chapter. The function Θ is a metric on the set of all linear closed subspaces of X, see [**71**, §2, Chapter IV] and the convergence $M_n \to M$ is obviously defined by $\Theta(M_n, M) \to 0$ as $n \to \infty$.

Theorem 4.26. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies Browder's theorem;*

(ii) *the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta(T)$ in the gap metric;*

(iii) *the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta(T)$;*

(iv) *the mapping $\lambda \to (\lambda I - T)(X)$ is not continuous at every point $\lambda \in \Delta(T)$ in the gap metric.*

Proof

(i) $\Rightarrow$ (ii) By Theorem 4.25 if T satisfies Browder's theorem then $\Delta(T) \subseteq \operatorname{iso} \sigma(T)$. For every $\lambda_0 \in \Delta(T)$ we have $\alpha(\lambda_0 I - T) > 0$ and since λ_0 is an isolated point of $\sigma(T)$ there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at λ_0 in the gap metric.

(ii) $\Rightarrow$ (i) Let $\lambda_0 \in \Delta(T)$ be arbitrary. By the punctured neighborhood theorem there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that, $\lambda I - T \in \Phi(X)$

for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$, $\alpha(\lambda I - T)$ is constant as λ ranges on $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$,

$$\text{ind}(\lambda I - T) = \text{ind}(\lambda_0 I - T) \quad \text{for all} \quad \lambda \in \mathbb{D}(\lambda_0, \varepsilon),$$

and

$$0 \leq \alpha(\lambda I - T) \leq \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon).$$

The discontinuity of the mapping $\lambda \to \ker(\lambda I - T)$ at every $\lambda \in \Delta(T)$ implies that

$$0 \leq \alpha(\lambda I - T) < \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}.$$

We claim that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. To see this, suppose that there is $\lambda_1 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ such that $\alpha(\lambda_1 I - T) > 0$. Clearly, $\lambda_1 \in \Delta(T)$, so arguing as for λ_0 we obtain a $\lambda_2 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0, \lambda_1\}$ such that

$$0 < \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T),$$

and this is impossible since $\alpha(\lambda I - T)$ is constant for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore $0 = \alpha(\lambda I - T)$ for $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, and since $\lambda I - T \in W(X)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$ we conclude that $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Hence $\lambda_0 \in \text{iso}\, \sigma(T)$, thus T satisfies Browder's theorem by Theorem 4.25.

To show the equivalences of the assertions (ii), (iii) and (iv) observe first that since for every $\lambda_0 \in \Delta(T)$ we have $\lambda_0 I - T \in \Phi(X)$ and hence the range $(\lambda I - T)(X)$ is closed for all λ near to λ_0. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) then follow from Theorem 1.38 of [**1**]. ∎

Browder's theorem may be also characterized by means of the Saphar spectrum, defined as

$$\sigma_{\text{sa}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Saphar}\}.$$

Theorem 4.27. *For a bounded operator T each of the following statements is equivalent to Browder's theorem:*

(i) $\Delta(T) \subseteq \sigma_{\text{se}}(T)$;

(ii) $\Delta(T) \subseteq \text{iso}\, \sigma_{\text{se}}(T)$;

(iii) $\Delta(T) \subseteq \sigma_{\text{sa}}(T)$;

(iv) $\Delta(T) \subseteq \text{iso}\, \sigma_{\text{sa}}(T)$.

Proof By Theorem 1.38 of [**1**] the equivalent conditions of Theorem 4.26 are equivalent to saying that $\lambda I - T$ is not semi-regular for all $\lambda \in \Delta(T)$.

(i) $\Leftrightarrow$ (ii) The implication (ii) $\Rightarrow$ (i) is obvious. To show that (i) $\Rightarrow$ (ii) suppose that $\Delta(T) \subseteq \sigma_{\rm se}(T)$. If $\lambda_0 \in \Delta(T)$ then $\lambda_0 I - T \in \Phi_+(X)$ so $\lambda_0 I - T$ is essentially semi-regular, in particular of Kato type. By Theorem 1.64 there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. But $\lambda_0 \in \sigma_{\rm se}(T)$, so $\lambda_0 \in \operatorname{iso} \sigma_{\rm se}(T)$.

(i) $\Leftrightarrow$ (iii) The implication (i) $\Rightarrow$ (iii) is immediate, since $\sigma_{\rm se}(T) \subseteq \sigma_{\rm sa}(T)$.

To show the implication (iii) $\Rightarrow$ (i) suppose that $\Delta(T) \subseteq \sigma_{\rm sa}(T)$. Let $\lambda \in \Delta(T)$. Then $\alpha(\lambda I - T) < \infty$ and since $\lambda I - T \in W(X)$ it follows that $\beta(\lambda I - T) < \infty$. Clearly, $\ker(\lambda I - T)$ is complemented, since it is finite-dimensional, and $(\lambda I - T)(X)$ is complemented, since it is closed and finite-codimensional. Therefore T admits a generalized inverse, by Theorem 1.10, and from $\lambda \in \sigma_{\rm sa}(T)$ it then follows that $\lambda I - T$ is not semi-regular. Thus $\Delta(T) \subseteq \sigma_{\rm se}(T)$.

(iv) $\Rightarrow$ (iii) Obvious.

(ii) $\Rightarrow$ (iv) Let $\lambda_0 \in \Delta(T)$. Since $\lambda_0 I - T \in W(X)$, it there exists an open disc $\mathbb{D}$ centered at λ_0 such that $\lambda I - T \in W(X)$ for all $\lambda \in \mathbb{D}$, so $\lambda I - T$ is Fredholm and hence admits a generalized inverse for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. On the other hand, λ_0 is isolated in $\sigma_{\rm se}(T)$, so $\lambda_0 \in \operatorname{iso} \sigma_{\rm sa}(T)$. ∎

We now establish some characterization of operators satisfying Browder's theorem in terms of the quasi-nilpotent parts $H_0(\lambda I - T)$.

Theorem 4.28. *For a bounded operator $T \in L(X)$ the following statements are equivalent;*

(i) *Browder's theorem holds for T;*

(ii) *$H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta(T)$;*

(iii) *$H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta(T)$;*

(iv) *$K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta(T)$.*

Proof (i) $\Leftrightarrow$ (ii) Suppose that T satisfies Browder's theorem. By Theorem 4.23 then

$$\Delta(T) = p_{00}(T) = \sigma(T) \setminus \sigma_{\rm b}(T).$$

If $\lambda \in \Delta(T)$ then $\lambda I - T \in B(X)$, so λ is isolated in $\sigma(T)$ and hence T has SVEP at λ. We also have that $\lambda I - T \in \Phi(X)$, so, from Theorem 2.47 we conclude that $H_0(\lambda I - T)$ is finite-dimensional.

Conversely, suppose that $H_0(\lambda I - T)$ is finite-dimensional for every

$\lambda \in \Delta(T)$. By Theorem 2.45 then T has SVEP at every $\lambda \in \Delta(T)$ and since $\lambda I - T \in W(X)$ we also have, again by Theorem 2.45, that $p(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ it then follows that $q(\lambda I - T) < \infty$, hence $\lambda I - T \in B(X)$ for all $\lambda \in \Delta(T)$. Hence $\lambda \notin \sigma_{\rm b}(T)$.

On the other hand, $\Delta(T) \subseteq \sigma(T)$, so $\Delta(T) \subseteq p_{00}(T)$. By Lemma 4.20, it follows that $\Delta(T) = p_{00}(T)$, and by Theorem 4.23 we then conclude that T satisfies Browder's theorem.

(ii) $\Leftrightarrow$ (iii) is clear by Theorem 2.47.

To show the equivalence (ii) $\Leftrightarrow$ (iv) observe that, by Theorem 4.25, Browder's theorem is equivalent to saying that every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$.

The equivalence (ii) $\Leftrightarrow$ (iv) follows by Theorem 2.9. ■

Browder's theorem is preserved under some commuting perturbations:

Theorem 4.29. *If $T \in L(X)$, R is a Riesz operator commuting with T, then T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem.*

Proof By Theorem 4.7 $\sigma_{\rm w}(T)$ and $\sigma_{\rm b}(T)$ are invariant under Riesz commuting perturbations. ■

4. a-Browder's theorem

An approximation point version of Browder's theorem is given by the so-called a-Browder's theorem.

Definition 4.30. *A bounded operator $T \in L(X)$ is said to satisfy* a-Browder's theorem *if*

$$\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T),$$

or equivalently, by Theorem 4.5, if

$$acc\, \sigma_{\rm a}(T) \subseteq \sigma_{\rm uw}(T).$$

Define

$$p_{00}^a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T) = \{\lambda \in \sigma_{\rm a}(T) : \lambda I - T \in \mathcal{B}_+(X)\}.$$

Let

$$\Delta_a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T).$$

Since $\lambda I - T \in W_+(X)$ implies that $(\lambda I - T)(X)$ is closed, we can write

$$\Delta_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W_+(X),\, 0 < \alpha(\lambda I - T)\}.$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$.

Lemma 4.31. *For every $T \in L(X)$ we have*

(i) $p_{00}^a(T) \subseteq \pi_{00}^a(T)$. *In particular, every $\lambda \in p_{00}^a(T)$ is an isolated point of $\sigma_{\rm a}(T)$.*

(ii) $p_{00}^a(T) \subseteq \Delta_a(T) \subseteq \sigma_{\rm a}(T)$.

Proof (i) If $\lambda \in p_{00}^a(T)$ then $\lambda I - T \in \Phi_+(X)$ and $p(\lambda I - T) < \infty$. Since T has SVEP at λ this implies that λ is isolated in $\sigma_{\rm a}(T)$. Furthermore, $0 < \alpha(\lambda I - T) < \infty$, since $\lambda I - T \in B_+(X)$ has closed range and $\lambda \in \sigma_{\rm a}(T)$. Therefore, $\lambda \in \pi_{00}^a(T)$.

(ii) The inclusion $p_{00}^a(T) \subseteq \Delta_a(T)$ is immediate, since $B_+(X) \subseteq W_a(X)$, $\lambda I - T$ has closed range and $\lambda \in \sigma_{\rm a}(T)$. The inclusion $\Delta_a(T) \subseteq \sigma_{\rm a}(T)$ is obvious. ∎

Theorem 4.32. *For a bounded operator $T \in L(X)$, a-Browder's theorem holds for T if and only if $p_{00}^a(T) = \Delta_a(T)$. In particular, a-Browder's theorem holds whenever $\Delta_a(T) = \emptyset$.*

Proof Suppose that T satisfies a-Browder's theorem. Clearly, the equality $p_{00}^a(T) = \Delta_a(T)$ holds whenever $\Delta_a(T) = \emptyset$. Suppose then $\Delta_a(T) \neq \emptyset$ and let $\lambda \in \Delta_a(T)$. Then $\lambda I - T \in W_a(X)$ and $\lambda \in \sigma_{\rm a}(T)$. From the equality $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$ it then follows that $\lambda I - T \in B_+(X)$, so $\lambda \in p_{00}^a(T)$. Hence $\Delta_a(T) \subseteq p_{00}^a(T)$, and hence $p_{00}^a(T) = \Delta_a(T)$.

Conversely, suppose that $p_{00}^a(T) = \Delta_a(T)$. Let $\lambda \notin \sigma_{\rm uw}(T)$. We show that $\lambda \notin \sigma_{\rm ub}(T)$. If $\lambda \notin \operatorname{acc} \sigma_{\rm a}(T)$ then by part (i) of Theorem 4.5 we have $\lambda \notin \sigma_{\rm ub}(T)$. Consider the other case $\lambda \in \operatorname{acc} \sigma_{\rm a}(T)$. Since $\sigma_{\rm a}(T)$ is closed then $\lambda \in \sigma_a(T)$ and $(\lambda I - T)(X)$ being closed it must be $0 < \alpha(\lambda I - T)$. Moreover, $\lambda I - T \in W_+(X)$ and hence $\lambda \in \Delta_a(T) = p_{00}^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T)$, thus $\lambda \notin \sigma_{\rm ub}(T)$. Therefore we have $\sigma_{\rm ub}(T) \subseteq \sigma_{\rm uw}(T)$ and since the reverse inclusion is satisfied by every operator we then conclude that $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$, i.e. T satisfies a-Browder's theorem.

The last assertion is clear. ∎

Theorem 4.33. *Suppose that either T or T^* has SVEP. Then a-Browder theorem holds for $f(T)$ and $f(T)^*$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof Either $f(T)$ or $f(T)^*$ has SVEP, by Theorem 4.18, and by Theorem 4.6 we then have $\sigma_{\rm ub}(f(T)) = \sigma_{\rm uw}(f(T))$, so $f(T)$ satisfies a-Browder's theorem. a-Browder's for $f(T)^*$ follows by duality and by Theorem 4.6. ∎

A precise description of operators satisfying a-Browder's theorem may be also given in terms of SVEP at certain sets.

Theorem 4.34. *If $T \in L(X)$ the following statements hold:*

(i) *T satisfies a-Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$.*

(ii) *T^* satisfies a-Browder's theorem if and only if T^* has SVEP at every $\lambda \notin \sigma_{\rm lw}(T)$.*

(iii) *If T has SVEP at every $\lambda \notin \sigma_{\rm lw}(T)$ then a-Browder's theorem holds for T^*.*

(iv) *If T^* has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$ then a-Browder's theorem holds for T.*

Proof (i) Suppose that $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$. If $\lambda \notin \sigma_{\rm uw}(T)$ then $\lambda I - T \in B_+(X)$ so $p(\lambda I - T) < \infty$ and hence T has SVEP at λ. Conversely, if T has SVEP at every point which is not in $\sigma_{\rm uw}(T)$, then for every $\lambda \notin \sigma_{\rm uw}(T)$, $\lambda I - T \in \Phi_+(X)$ and the SVEP at λ by Theorem 2.45 implies that $p(\lambda I - T) < \infty$, and hence $\lambda \notin \sigma_{\rm ub}(T)$. Therefore $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$.

(ii) Obvious, since $\sigma_{\rm lw}(T) = \sigma_{\rm uw}(T^*)$.

(iii) Suppose that T has SVEP at every point which does not belong to $\sigma_{\rm lw}(T)$. If $\lambda \notin \sigma_{\rm uw}(T^*) = \sigma_{\rm lw}(T)$ then $\lambda I - T \in \Phi_-(X)$ with $\text{ind}(\lambda I - T) \geq 0$. By Theorem 2.45 the SVEP of T at λ entails that $p(\lambda I - T) < \infty$ and hence by Theorem 1.21 we have $\text{ind}(\lambda I - T) \leq 0$. Therefore, $\text{ind}(\lambda I - T) = 0$, and since $p(\lambda I - T) < \infty$ we conclude by Theorem 1.21 that $q(\lambda I - T) < \infty$, so $\lambda \notin \sigma_{\rm lb}(T) = \sigma_{\rm ub}(T^*)$. Thus $\sigma_{\rm ub}(T^*) \subseteq \sigma_{\rm uw}(T^*)$ and since the reverse inclusion holds for every operator we conclude that $\sigma_{\rm ub}(T^*) = \sigma_{\rm uw}(T^*)$.

(iv) If $\lambda \notin \sigma_{\rm uw}(T)$ then $\lambda I - T \in W_+(X)$ and hence $\text{ind}(\lambda I - T) \leq 0$. Since $\lambda I - T \in \Phi_+(X)$ the SVEP of T^* at λ implies that $q(\lambda I - T) < \infty$ and hence by Theorem 1.21 we have $\text{ind}(\lambda I - T) \geq 0$. Therefore $\text{ind}(\lambda I - T) = 0$, and since $q(\lambda I - T) < \infty$ it then follows that also $p(\lambda I - T)$ is finite, see Theorem 1.21. Consequently, $\lambda \notin \sigma_{\rm ub}(T)$ from which we conclude that $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$. ∎

Since $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$, from Theorem 4.34 and Theorem 4.23 we readily obtain:

$$a\text{-Browder's theorem for } T \Rightarrow \text{Browder's theorem for } T.$$

Note that the reverse of the assertions (iii) and (iv) of Theorem 4.23 generally do not hold. An example of unilateral weighted shifts T on $\ell^p(\mathbb{N})$ for which a-Browder's theorem holds for T (respectively, a-Browder's theorem holds for T^*) and such that SVEP fails at some points $\lambda \notin \sigma_{\mathrm{lw}}(T)$ (respectively, at some points $\lambda \notin \sigma_{\mathrm{uw}}(T)$) may be found in [**9**].

The following results are analogous to the results of Theorem 4.25 and Theorem 4.26, and give a precise spectral picture of operator satysfing a-Browder's theorem.

Theorem 4.35. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies a-Browder's theorem;*

(ii) *$\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$;*

(iii) *$\Delta_a(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$, $\partial\sigma_{\mathrm{a}}(T)$ the topological boundary of $\sigma_{\mathrm{a}}(T)$;*

(iv) *the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta_a(T)$ in the gap metric;*

(v) *the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta_a(T)$;*

(vi) *the mapping $\lambda \to (\lambda I - T)(X)$ is not continuous at every $\lambda \in \Delta_a(T)$ in the gap metric;*

(vii) *$\Delta_a(T) \subseteq \sigma_{\mathrm{se}}(T)$;*

(viii) *$\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{se}}(T)$;*

(ix) *$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso}\sigma_{\mathrm{a}}(T)$.*

Proof The equivalences are obvious if $\Delta_a(T) = \emptyset$, so we may suppose that $\Delta_a(T)$ is non-empty.

(i) $\Leftrightarrow$ (ii) By Theorem 4.32 if T satisfies a-Browder's theorem then $\Delta_a(T) = p_{00}^a(T)$, so by Lemma 4.31, part (i), every $\lambda \in \Delta_a(T)$ is an isolated point of $\sigma_{\mathrm{a}}(T)$. Conversely, suppose that $\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and take $\lambda \in \Delta_a(T)$. Then T has SVEP at λ, since λ is an isolated point of $\sigma_{\mathrm{a}}(T)$, and being $\lambda I - T \in \Phi_+(X)$ the SVEP at λ is equivalent to saying that $p(\lambda I - T) < \infty$, and hence $\lambda I - T \in B_+(X)$. Therefore, $\lambda \in p_{00}^a(T)$, from which we conclude that $\Delta_a(T) = p_{00}^a(T)$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (ii) Suppose that the inclusion $\Delta_a(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$ holds. Let $\lambda_0 \in \Delta_a(T)$ be arbitrary given. We show that T has SVEP at λ_0. Let $f : U \to X$ be an analytic function defined on an open disc U of λ_0 which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$. Since $\lambda_0 \in \partial\sigma_{\mathrm{a}}(T)$ we can choose $\mu \neq \lambda_0$, $\mu \in U$ such that $\mu \notin \sigma_{\mathrm{a}}(T)$. Consider an open disc W of μ such that $W \subseteq U$. We know that T has SVEP at μ, so $f(\lambda) = 0$ for all $\lambda \in W$. The identity theorem for analytic functions then implies that $f(\lambda) = 0$ for all $\lambda \in U$, hence T has SVEP at λ_0. Finally, $\lambda_0 I - T \in \Phi_+(X)$, since $\lambda_0 \in \Delta_a(T)$. The SVEP at λ_0 then implies that $\sigma_{\mathrm{a}}(T)$ does not cluster at λ_0, and $\Delta_a(T)$ being a subset of $\sigma_{\mathrm{a}}(T)$ we then conclude that $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$.

(ii) $\Rightarrow$ (iv) Suppose that $\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$. For every $\lambda_0 \in \Delta_a(T)$ then $\alpha(\lambda_0 I - T) > 0$ and since $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at λ_0.

(iv) $\Rightarrow$ (ii) Let $\lambda_0 \in \Delta_a(T)$ be arbitrary. By Theorem 1.58 there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\alpha(\lambda I - T)$ is constant as λ ranges on $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $\lambda I - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$,

$$\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda_0 I - T) \quad \text{for all} \quad \lambda \in \mathbb{D}(\lambda_0, \varepsilon),$$

and

$$0 \leq \alpha(\lambda I - T) \leq \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon).$$

Since the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at λ_0 it then follows that

$$0 \leq \alpha(\lambda I - T) < \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}.$$

We claim that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. To see this, suppose that there is $\lambda_1 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ such that $\alpha(\lambda_1 I - T) > 0$. From $\operatorname{ind}(\lambda_1 I - T) = \operatorname{ind}(\lambda_0 I - T) \leq 0$ we see that $\lambda I - T \in W_+(T)$ and hence $\lambda_1 \in \Delta_a(T)$. Repeating the same reasoning as above we may choose a $\lambda_2 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0, \lambda_1\}$ such that

$$0 < \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T)$$

and this is impossible since $\alpha(\lambda I - T)$ is constant for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore $\alpha(\lambda I - T) = 0$ for $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ and since $(\lambda I - T)(X)$ is closed for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$ we can conclude that $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$, as desired.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) To show these equivalences observe first that for every $\lambda \in \Delta_a(T)$ the range $(\lambda I - T)(X)$ is closed. The equivalences then follow from Theorem 1.38 of [**1**].

(vi) $\Leftrightarrow$ (vii) If $\lambda_0 \in \Delta_a(T)$ then there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ centered at λ_0 such that $\lambda I - T$ has closed range for all $\lambda \in \mathbb{D}(\lambda, \varepsilon)$. The equivalence (vi) $\Leftrightarrow$ (vii) then easily follows from Theorem 1.38 of [**1**].

(viii) $\Rightarrow$ (vii) Clear.

(vii) $\Rightarrow$ (viii) Suppose that $\Delta_a(T) \subseteq \sigma_{\rm se}(T)$. If $\lambda_0 \in \Delta_a(T)$ then $\lambda_0 I - T \in \Phi_+(X)$ so $\lambda_0 I - T$ is essentially semi-regular, in particular of Kato type. By Theorem 1.64 then there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ centered at λ_0 such that $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. But $\lambda_0 \in \sigma_{\rm se}(T)$, so $\lambda_0 \in \operatorname{iso} \sigma_{\rm se}(T)$.

(i) $\Leftrightarrow$ (ix) The inclusion $\sigma_{\rm uw}(T) \cup \operatorname{iso} \sigma_{\rm a}(T) \subseteq \sigma_{\rm a}(T)$ holds for every $T \in L(X)$, so we need only to prove the reverse inclusion. Suppose that a-Browder's theorem holds. If $\lambda \in \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T)$ then, Theorem 4.34 T has SVEP at λ, and hence by Theorem 2.51 $\lambda \in \operatorname{iso} \sigma_{\rm a}(T)$. Therefore $\sigma_{\rm a}(T) \subseteq \sigma_{\rm uw}(T) \cup \operatorname{iso} \sigma_{\rm a}(T)$, so the equality (ix) is proved.

Conversely, suppose that $\sigma_{\rm a}(T) = \sigma_{\rm uw}(T) \cup \operatorname{iso} \sigma_{\rm a}(T)$. Let $\lambda \notin \sigma_{\rm uw}(T)$. There are two possibilities: $\lambda \in \operatorname{iso} \sigma_{\rm a}(T)$ or $\lambda \notin \operatorname{iso} \sigma_{\rm a}(T)$. By if $\lambda \in \operatorname{iso} \sigma_{\rm a}(T)$ then T has SVEP at λ. In the other case $\lambda \notin \sigma_{\rm uw}(T) \cup \operatorname{iso} \sigma_{\rm a}(T) = \sigma_{\rm a}(T)$, and hence T has SVEP at λ. From Theorem 4.34 we then conclude that a-Browder's theorem holds for T.

The second assertion follows by duality, since $\sigma_{\rm s}(T) = \sigma_{\rm a}(T^*)$ and $\sigma_{\rm lw}(T) = \sigma_{\rm uw}(T^*)$ for every $T \in L(X)$. ∎

Remark 4.36. It should be noted that if $T \in \Phi_+(X)$, by Theorem 1.64 the property that T is not semi-regular may be expressed by saying that the *jump* $j(T)$ is greater than 0, so

$$a\text{-Browder's theorem holds for } T \Leftrightarrow j(\lambda I - T) > 0 \quad \text{for all } \lambda \in \Delta_a(T).$$

Corollary 4.37. *Suppose that T^* has SVEP. Then $\Delta_a(T) \subseteq \operatorname{iso} \sigma(T)$.*

Proof Also here we can suppose that $\Delta_a(T)$ is non-empty. If T^* has SVEP then a-Browder' s theorem holds for T, so by Theorem 4.26 $\Delta_{\rm a} \subseteq \operatorname{iso} \sigma_{\rm a}(T)$. Moreover, by Corollary 2.48, for all $\lambda \in \Delta_{\rm a}(T)$ we have $\operatorname{ind}(\lambda I - T) \leq 0$, so $0 < \alpha(\lambda I - T) \leq \beta(\lambda I - T)$, and hence $\lambda \in \sigma_{\rm s}(T)$. Now, if $\lambda \in \Delta_{\rm a}(T)$ the SVEP for T^* entails by Theorem 2.52 that $\lambda \in \operatorname{iso} \sigma_{\rm s}(T)$, and hence $\lambda \in \operatorname{iso} \sigma_{\rm s}(T) \cap \operatorname{iso} \sigma_{\rm a}(T) = \operatorname{iso} \sigma(T)$. ∎

We now give a further characterization of operators satisfying a-Browder's theorem in terms of the quasi-nilpotent part $H_0(\lambda I - T)$.

Theorem 4.38. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *a-Browder's theorem holds for T.*

(ii) *$H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta_a(T)$.*

(iii) *$H_0(\lambda I - T)$ is closed for every $\lambda \in \Delta_a(T)$.*

(iv) *$\mathcal{N}^\infty(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta_a(T)$.*

(v) *$\mathcal{N}^\infty(\lambda I - T)$ is closed for every $\lambda \in \Delta_a(T)$.*

Proof There is nothing to prove if $\Delta_a(T) = \emptyset$. Suppose that $\Delta_a(T) \neq \emptyset$.

(i) $\Leftrightarrow$ (ii) Suppose that T satisfies a-Browder's theorem. By Theorem 4.32 then

$$\Delta_a(T) = p_{00}^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T).$$

If $\lambda \in \Delta_a(T)$ then λ is isolated in $\sigma_{\rm a}(T)$ and hence T has SVEP at λ. We also have that $\lambda I - T \in \Phi_+(X)$, so, from Theorem 2.47 we conclude that $H_0(\lambda I - T)$ is finite-dimensional.

Conversely, suppose that $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta_a(T)$. To show that T satisfies a-Browder's theorem it suffices to prove that T has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$. Since T has SVEP at every $\lambda \notin \sigma_{\rm a}(T)$ we can suppose that $\lambda \in \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \Delta_a(T)$. Since $\lambda I - T \in \Phi_+(X)$ the SVEP at λ it then follows by Theorem 2.47.

(ii) $\Leftrightarrow$ (iii) Since $\lambda I - T \in \Phi_+(X)$ for every $\lambda \in \Delta_a(T)$, the equivalence follows from Theorem 2.47.

(ii) $\Rightarrow$ (iv) Clear, since $\mathcal{N}^\infty(\lambda I - T) \subseteq H_0(\lambda I - T)$.

(iv) $\Rightarrow$ (i) Also here we prove that T has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$. We can suppose that $\lambda \in \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \Delta_a(T)$, since the SVEP is satisfied at every point $\mu \notin \sigma_{\rm a}(T)$. By assumption $\mathcal{N}^\infty(\lambda I - T)$ is finite-dimensional and from the inclusion $\ker(\lambda I - T)^n \subseteq \ker(\lambda I - T)^{n+1} \subseteq \mathcal{N}^\infty(\lambda I - T)$, $n \in \mathbb{N}$, it is evident that there exists $p \in \mathbb{N}$ such that $\ker(\lambda I - T)^p = \ker(\lambda I - T)^{p+1}$. Hence $p(\lambda I - T) < \infty$, so by Theorem 2.39 T has SVEP at λ.

(iv) $\Rightarrow$ (v) Obvious.

(v) $\Rightarrow$ (i) As above it suffices to prove that $p(\lambda I - T) < \infty$ for every $\lambda \notin \sigma_{\rm uw}(T)$. We use a standard argument from the well-known Baire theorem. Suppose $p(\lambda I - T) = \infty$, $\lambda \notin \sigma_{\rm uw}(T)$. By assumption

$\mathcal{N}^\infty(\lambda I - T) = \bigcup_{n=1}^\infty \ker(\lambda I - T)^n$ is closed so it is of second category in itself. Moreover, $\ker(\lambda I - T)^n \neq \mathcal{N}^\infty(\lambda I - T)$ implies that $\ker(\lambda I - T)^n$ is of the first category as subset of $\mathcal{N}^\infty(\lambda I - T)$ and hence also $\mathcal{N}^\infty(\lambda I - T)$ is of the first category. From this it then follows that $\mathcal{N}^\infty(\lambda I - T)$ is not closed, a contradiction. Therefore $p(\lambda I - T) < \infty$ for every $\lambda \notin \sigma_{\mathrm{uw}}(T)$. ∎

Theorem 4.39. *If $K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta_a(T)$ then a-Browder's theorem holds for T.*

Proof We show that T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$. We can suppose that $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \Delta_a(T)$. By assumption $K(\lambda I - T)$ is finite-codimensional and hence, by Theorem 2.47, $q(\lambda I - T) < \infty$, from which it follows that $\mathrm{ind}\,(\lambda I - T) \geq 0$, see Theorem 1.21. On the other hand, $\lambda I - T \in W_+(X)$, so $\mathrm{ind}\,(\lambda I - T) \leq 0$, from which we obtain that $\mathrm{ind}\,(\lambda I - T) = 0$. Again by Theorem 1.21 we conclude that $p(\lambda I - T) < \infty$, and hence T has SVEP at λ. ∎

5. Weyl's theorem

In this section we shall introduce an important property shared by several classes of operators. This property has been first observed by Weyl for normal operators on Hilbert spaces [**105**]. To define this property , for a bounded operator $T \in L(X)$ define

$$\pi_{00}(T) := \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi_{00}^a(T) := \{\lambda \in \mathrm{iso}\,\sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Lemma 4.40. *For every $T \in L(X)$ we have*

$$p_{00}(T) \subseteq p_{00}^a(T) \quad \textit{and} \quad \pi_{00}(T) \subseteq \pi_{00}^a(T). \tag{43}$$

Proof If $\lambda \in p_{00}(T)$ then λ is an isolated point of $\sigma(T)$. Moreover, $\lambda \in \sigma_{\mathrm{a}}(T)$ since $\alpha(\lambda I - T) > 0$ (in fact, if were $\alpha(\lambda I - T) = 0$, we would have $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, see Theorem 1.21, and hence $\lambda \notin \sigma(T)$, a contradiction). Therefore λ is an isolated point of $\sigma_{\mathrm{a}}(T)$, so the inclusion $p_{00}(T) \subseteq p_{00}^a(T)$ is proved. The inclusion $\pi_{00}(T) \subseteq \pi_{00}^a(T)$ is obvious. ∎

The following concept has been introduced by Coburn [**41**].

Definition 4.41. *A bounded operator $T \in L(X)$ is said to satisfy* Weyl's theorem *if*

$$\Delta(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T).$$

This equality is the property proved by Weyl in the case where T is a normal operator on a Hilbert space. In this section we shall extend this theorem to several other classes of operators.

Theorem 4.42. *If a bounded operator $T \in L(X)$ satisfies Weyl's theorem then*

$$p_{00}(T) = \pi_{00}(T) = \Delta(T).$$

Proof Suppose that T satisfies Weyl's theorem. By definition then $\Delta(T) = \pi_{00}(T)$. We show now the equality $p_{00}(T) = \pi_{00}(T)$. It suffices to prove the inclusion $\pi_{00}(T) \subseteq p_{00}(T)$. Let λ be an arbitrary point of $\pi_{00}(T)$. Since λ is isolated in $\sigma(T)$ then T has SVEP at λ and from the equality $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$ we know that $\lambda I - T \in W(X)$. Hence $\lambda I - T \in \Phi(X)$ and the SVEP at λ by Theorem 2.47 implies that $p(\lambda I - T) < \infty$, so $\lambda \in p_{00}(T)$. ∎

The condition $\pi_{00}(T) = p_{00}(T)$ may be formulated in several equivalent ways:

Theorem 4.43. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $\pi_{00}(T) = p_{00}(T)$;

(ii) $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \emptyset$;

(iii) $\sigma_{\mathrm{sf}}(T) \cap \pi_{00}(T) = \emptyset$;

(iv) $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi_{00}(T)$;

(v) $H_0(\lambda I - T)$ *is finite-dimensional for all* $\lambda \in \pi_{00}(T)$;

(vi) $K(\lambda I - T)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;

(vii) $(\lambda I - T)^{\infty}(X)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;

(viii) $\beta(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$;

(ix) $q(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$;

(x) *The mapping* $\lambda \to \gamma(\lambda I - T)$ *is not continuous at each* $\lambda_0 \in \pi_{00}(T)$.

Proof (i) ⇒ (ii) We have $p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, so $\sigma_{\mathrm{b}}(T) \cap p_{00}(T) = \emptyset$ and this obviously implies $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \emptyset$, since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$.

(ii) ⇒ (iii) Obvious, since $\sigma_{\mathrm{sf}}(T) \subseteq \sigma_{\mathrm{w}}(T)$.

(iii) $\Rightarrow$ (iv) If $\lambda \in \pi_{00}(T)$ then $\lambda I - T$ is semi-Fredholm, hence $(\lambda I - T)(X)$ is closed.

(iv) $\Rightarrow$ (v) Let $\lambda \in \pi_{00}(T)$. If $(\lambda I - T)(X)$ is closed then $\lambda I - T \in \Phi_+(X)$. Since T has SVEP at every isolated point of $\sigma(T)$, by Theorem 2.47 then $H_0(\lambda I - T)$ is finite-dimensional.

(v) $\Rightarrow$ (vi) If $\lambda \in \operatorname{iso} \sigma(T)$, then by Theorem 2.9 $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$. Hence, if $H_0(\lambda I - T)$ is finite-dimensional then $K(\lambda I - T)$ is finite-codimensional.

(vi) $\Rightarrow$ (vii) Immediate, since $K(\lambda I - T) \subseteq (\lambda I - T)^\infty(X)$ for every $\lambda \in \mathbb{C}$.

(vii) $\Rightarrow$ (viii) $(\lambda I - T)^\infty(X) \subseteq (\lambda I - T)(X)$ for every $\lambda \in \mathbb{C}$.

(viii) $\Rightarrow$ (i) For every $\lambda \in \pi_{00}(T)$ we have $\alpha(\lambda I - T) < \infty$, and hence if $\beta(\lambda I - T) < \infty$ then $\lambda I - T \in \Phi(X)$. Since $\lambda \in \operatorname{iso} \sigma(T)$, by Theorem 2.45 the SVEP of both T and T^* at λ ensures that $p(\lambda I - T) = q(\lambda I - T) < \infty$. Therefore $\pi_{00}(T) \subseteq p_{00}(T)$, and since the opposite inclusion is satisfied by every operator it then follows that $\pi_{00}(T) = p_{00}(T)$.

(i) $\Rightarrow$ (ix) Clear.

(ix) $\Rightarrow$ (viii) If $q(\lambda I - T) < \infty$ by Theorem 1.21 we have $\beta(\lambda I - T) \leq \alpha(\lambda I - T) < \infty$.

(iv) $\Leftrightarrow$ (x) Observe first that if $\lambda_0 \in \pi_{00}(T)$ there exists a punctured disc $\mathbb{D}_0$ centered at λ_0 such that

$$\gamma(\lambda I - T) \leq |\lambda - \lambda_0| \quad \text{for all } \lambda \in \mathbb{D}_0. \tag{44}$$

In fact, if λ_0 is isolated in $\sigma(T)$ then $\lambda I - T$ is invertible, and hence has closed range, in a punctured disc $\mathbb{D}$ centered at λ_0. Take $0 \neq x \in \ker(\lambda_0 I - T)$. Then

$$\begin{aligned} \gamma(\lambda I - T) &\leq \frac{\|(\lambda I - T)x\|}{\operatorname{dist}\,(x, \ker(\lambda I - T))} = \frac{\|(\lambda I - T)x\|}{\|x\|} \\ &= \frac{\|(\lambda I - T)x - (\lambda_0 I - T)x\|}{\|x\|} = |\lambda - \lambda_0|. \end{aligned}$$

Clearly, from the estimate (44) it follows $\gamma(\lambda I - T) \to 0$ as $\lambda \to \lambda_0$ so the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at a point $\lambda_0 \in \pi_{00}(T)$ if and only if $\gamma(\lambda_0 I - T) > 0$, or equivalently, $(\lambda_0 I - T)(X)$ is closed. Therefore the condition (iv) of Theorem 4.43 is equivalent to the condition (x). ∎

The following result shows the relationships between Browder's theorem and Weyl's theorem: Let us define

$$\Delta_{00}(T) := \Delta(T) \cup \pi_{00}(T).$$

Theorem 4.44. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T satisfies Weyl's theorem;*

(ii) *T satisfies Browder's theorem and $p_{00}(T) = \pi_{00}(T)$;*

(iii) *the map $\lambda \to \gamma(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta_{00}(T)$;*

(iv) *$H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \Delta_{00}(T)$;*

(v) *$K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta_{00}(T)$.*

Proof (i) $\Leftrightarrow$ (ii) The implication (i) $\Rightarrow$ (ii) is clear, from Theorem 4.42 and Theorem 4.23, while the implication (ii) $\Rightarrow$ (i) follows immediately from Theorem 4.23.

(i) $\Rightarrow$ (iii) By Theorem 4.42 we have that $\Delta_{00}(T) = \Delta(T)$ and T satisfies Browder's theorem. Therefore, by Theorem 4.26, the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta_{00}(T)$.

(iii) $\Rightarrow$ (ii) Suppose that $\lambda \to \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}(T) = \Delta(T) \cup \pi_{00}(T)$. The discontinuity at the points of $\Delta(T)$ entails, by Theorem 4.26, that T satisfies Browder's theorem, while the discontinuity at the points of $\pi_{00}(T)$ by Theorem 4.43 is equivalent to saying that $\pi_{00}(T) = p_{00}(T)$.

(i) $\Rightarrow$ (iv) If T satisfies Weyl's theorem then T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T) = \Delta(T)$, by Theorem 4.42. By Theorem 4.28 then $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \Delta_{00}(T) = \Delta(T)$.

(iv) $\Rightarrow$ (i) Since $\Delta(T) \subseteq \Delta_{00}(T)$ then Browder's theorem holds for T by Theorem 4.28. From $\pi_{00}(T) \subseteq \Delta_{00}(T)$ we know that $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \pi_{00}(T)$. Since every $\lambda \in \pi_{00}(T)$ is an isolated point of $\sigma(T)$ by Theorem 2.66 it then follows that $\lambda I - T$ is Browder. Therefore $\pi_{00}(T) = p_{00}(T)$, so by Theorem 4.23 T satisfies Weyl's theorem.

(iv) $\Leftrightarrow$ (v) By Theorem 2.9 for any point of $\lambda \in \pi_{00}(T)$ we have $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$. ■

¿From Theorem 4.26 and Theorem 4.44 we see that Browder's theorem and Weyl's theorem are equivalent to the discontinuity of the mapping $\lambda \to \gamma(\lambda I - T)$ at the points of two sets $\Delta(T)$ and $\Delta_{00}(T)$, respectively, with $\Delta_{00}(T)$ larger than $\Delta(T)$. Note that the discontinuity of the mapping $\lambda \to \ker(\lambda I - T)$ at every $\lambda \in \Delta_{00}(T) = \Delta(T) \cup \pi_{00}(T)$ does not imply Weyl's theorem. In fact, since every point of $\pi_{00}(T)$ is an isolated point of $\sigma(T)$, it is evident that the map $\lambda \to \ker(\lambda I - T)$ is not continuous at every $\lambda \in \pi^a_{00}(T)$ for *all* operators $T \in L(X)$.

Let $\mathcal{P}_0(X)$, X a Banach space, denote the class of all operators $T \in L(X)$ such that there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T). \tag{45}$$

We have seen in Theorem 4.43 that the conditions $p_{00}(T) = \pi_{00}(T)$ is equivalent to several other conditions. Another useful condition is given by the following result.

Theorem 4.45. *$T \in \mathcal{P}_0(X)$ if and only if $p_{00}(T) = \pi_{00}(T)$. In particular, if T has SVEP then Weyl's theorem holds for T if and only if $T \in \mathcal{P}_0(X)$.*

Proof Suppose $T \in \mathcal{P}_0(X)$ and $\lambda \in \pi_{00}(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since λ is isolated in $\sigma(T)$ then, by Theorem 2.9

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T),$$

so $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$ which implies, by Theorem 1.26, that $p(\lambda I - T) = q(\lambda I - T) \le p$. By definition of $\pi_{00}(T)$ we know that $\alpha(\lambda I - T) < \infty$ and this implies by Theorem 1.21 that $\beta(\lambda I - T) < \infty$. Therefore $\lambda \in p_{00}(T)$ and hence $\pi_{00}(T) \subseteq p_{00}(T)$. Since the opposite inclusion holds for every operator we then conclude that $p_{00}(T) = \pi_{00}(T)$.

Conversely, if $p_{00}(T) = \pi_{00}(T)$ and $\lambda \in \pi_{00}(T)$ then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. By Theorem 2.45 it then follows that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$.
The last assertion is clear from Theorem 4.44, since the SVEP entails Browder's theorem for T. ■

Theorem 4.46. *Suppose that $T \in L(X)$ and N is nilpotent such that $TN = NT$. Then $T \in \mathcal{P}_0(X)$ if and only if $T + N \in \mathcal{P}_0(X)$.*

Proof Suppose that $N^p = 0$. Observe first that without any assumption on T we have

$$\ker T \subseteq \ker (T+N)^p \quad \text{and} \quad \ker (T+N) \subseteq \ker T^p. \tag{46}$$

The first inclusion in (46) is clear, since for $x \in \ker T$ we have

$$(T+N)^p x = N^p x = 0.$$

To show the second inclusion in (46) observe that if $x \in \ker (T+N)$ then $T^p x = (-1)^p N^p x = 0$.

Suppose now that $T \in \mathcal{P}_0(X)$, or equivalently $p_{00}(T) = \pi_{00}(T)$. We show first that $\pi_{00}(T) = \pi_{00}(T+N)$. Let $\lambda \in \pi_{00}(T)$. There is no harm if we suppose $\lambda = 0$. From $\sigma(T+N) = \sigma(T)$ we see that $0 \in \text{iso}\,(T+N)$. Since $0 \in \pi_{00}(T)$ then $\alpha(T) > 0$ and hence by the first inclusion in (46) we have $\alpha(T+N)^p > 0$ and this obviously implies that $\alpha(T+N) > 0$. To show that $\alpha(T+N) < \infty$, observe that

$$\ker (T+N) \subseteq \ker T^p \subseteq H_0(T).$$

The equality $p_{00}(T) = \pi_{00}(T)$ by Theorem 4.43 is equivalent to saying that $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$, and hence $H_0(T)$ is finite-dimensional. Therefore $\alpha(T+N) < \infty$, so $0 \in \pi_{00}(T+N)$ and the inclusion $\pi_{00}(T) \subseteq \pi_{00}(T+N)$ is proved.

To show the opposite inclusion, assume that $0 \in \pi_{00}(T+N)$. Clearly, $0 \in \text{iso}\,\sigma(T) = \text{iso}\,\sigma(T+N)$. By assumption $\alpha(T+N) > 0$, so the second inclusion in (46) entails that $\alpha(T^p) > 0$ and this trivially implies that $\alpha(T) > 0$. We also have $\alpha(T+N) < \infty$ and hence, by Remark 2.78, $\alpha(T+N)^p < \infty$. From the first inclusion in (46) we then conclude that $\alpha(T) < \infty$. This shows that $0 \in \pi_{00}(T)$, so the equality $\pi_{00}(T) = \pi_{00}(T+N)$ is proved.

Finally, if $T \in \mathcal{P}_0(X)$ then $p_{00}(T+N) = p_{00}(T) = \pi_{00}(T) = \pi_{00}(T+N)$, so $T+N \in \mathcal{P}_0(X)$. Conversely, if $T+N \in \mathcal{P}_0(X)$ by symmetry we have

$$p_{00}(T) = p_{00}(T+N) = \pi_{00}(T+N) = \pi_{00}((T+N)-N) = \pi_{00}(T),$$

so the proof is complete. ∎

A large number of the commonly considered operators on Banach spaces and Hilbert spaces have SVEP and belong to the class $\mathcal{P}_0(X)$.

(a) A bounded operator $T \in L(X)$ on a Banach space X is said *paranormal* if

$$\|Tx\|^2 \leq \|T^2 x\| \|x\| \quad \text{for all } x \in X.$$

The operator $T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$. For every totally paranormal operator the following property $H(1)$ is satisfied

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \text{for all } \lambda \in \mathbb{C}, \tag{47}$$

see Aiena[**1**, Chap.3]. The class of totally paranormal operators includes all hyponormal operators on Hilbert spaces H. In the sequel denote by T' the Hilbert adjoint of $T \in L(H)$. The operator $T \in L(H)$ is said to be *hyponormal* if

$$\|T'x\| \le \|Tx\| \quad \text{for all } x \in X.$$

A bounded operator $T \in L(H)$ is said to be *quasi-hyponormal* if

$$\|T'Tx\| \le \|T^2x\| \quad \text{for all} \quad x \in H.$$

Also quasi-normal operators are totally paranormal, since these operators are hyponormal, see Conway [**43**].

An operator $T \in L(H)$ is said to be **-paranormal* if

$$\|T'x\|^2 \le \|T^2x\|$$

holds for all unit vectors $x \in H$. $T \in L(H)$ is said to be *totally *-paranormal* if $\lambda I - T$ is *-paranormal for all $\lambda \in \mathbb{C}$. Every totally $*$-paranormal operator satisfies property (47), see [**66**].

(b) The condition (47) is also satisfied by every injective p-hyponormal operator, see [**24**], where an operator $T \in L(H)$ on a Hilbert space H is said to be *p-hyponormal* , with $0 < p \le 1$, if $(T'T)^p \ge (TT')^p$.

(c) An operator $T \in L(H)$ is said to be *log-hyponormal* if T is invertible and satisfies $\log(T'T) \ge \log(TT')$. Every log-hyponormal operator satisfies the condition (47), see [**24**].

(d) A bounded operator $T \in L(X)$ is said to be *transaloid* if the spectral radius $r(\lambda I - T)$ is equal to $\|\lambda I - T\|$ for every $\lambda \in \mathbb{C}$. Every transaloid operator satisfies the condition (47), see Curto and Han [**44**].

(e) Given a Banach algebra A, a map $T : A \to A$ is said to be a *multiplier* if

$$(Tx)y = x(Ty) \quad \text{for all } x, y \in A.$$

For a commutative semi-simple Banach algebra A, let $M(A)$ denote the commutative Banach algebra of all multipliers, [**76**]. If $T \in M(A)$, A a commutative semi-simple Banach algebra, then $T \in L(A)$ and the condition (47) is satisfied, see [**10**]. In particular, this condition holds for every convolution operator on the group algebra $L^1(G)$, where G is a locally compact Abelian group, see Chapter 4 of Aiena [**1**].

(f) Every generalized scalar operator has SVEP, see [**76**], since it is decomposable. An operator similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*. The interested reader can find a well organized study of these operators in the Laursen and Neumann book [**76**]. The following class of operators has been introduced by Oudghiri [**88**].

Definition 4.47. *A bounded operator $T \in L(X)$ is said to have property $H(p)$ if*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{48}$$

for some $p = p(\lambda) \in \mathbb{N}$, see [**88**].

The condition $H(1)$ defined in (47), which corresponds to the case $p = p(\lambda) = 1$, is satisfied by all operators (a)–(e). Anyway the class $H(p)$ is strictly larger than the class $H(1)$. Indeed, Every generalized scalar operator satisfies the following property $H(p)$ and this captures the classes of operators listed in the next point (g)..

(g) An operator $T \in L(H)$ on a Hilbert space H is said to be M-*hyponormal* if there is $M > 0$ for which $TT' \leq MT'T$. M-hyponormal operators, p-hyponormal operators, log-hyponormal operators, and algebraically hyponormal operators are generalized scalars, so they satisfy the condition (48), see [**88**].

(h) An operator $T \in L(X)$ for which there exists a complex non constant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal.* If $T \in L(H)$ is algebraically paranormal then $T \in \mathcal{P}_0(H)$, see [**21**], but in general the condition (48) is not satisfied by paranormal operators, (for an example see [**21**, Example 2.3]). Note that if T is paranormal then T has SVEP, see [**21**], and this implies that also every algebraically paranormal operator has SVEP, see Theorem 2.40 of [**1**].

Theorem 4.48. *Suppose that T is any of the operators listed in (a)-(h). Then Weyl's theorem holds for T.*

Proof The condition (48), and in particular the condition (47), entails that T has SVEP. Weyl's theorem for the operators (a)-(g) then follows by Theorem 4.45. Also, every algebraically paranormal operators on

Hilbert spaces has SVEP, so Weyl's theorem for these operators follows again from Theorem 4.45. ∎

Weyl's theorem for algebraically paranormal operators on Hilbert spaces has been proved by Curto and Han [**45**]. It should be noted that the operators having property $H(p)$ we have much more: Weyl's theorem holds for $f(T)$ and $f(T^*)$ for every analytic function f defined on an open disc containing the spectrum, see Oudghiri [**88**]. The result of Theorem 4.48 may be improved as follows. Suppose that $T \in L(X)$ be *algebraic*, i.e. there exists a non-trivial polynomial q such that $q(T) = 0$. In [**89**] Oudghiri proved that if T has property (48) and K is an algebraic operators commuting with T then $T + K$ satisfies Weyl's theorem. This result has been extended in [**21**] to paranormal operators, i.e. if T is a paranormal operator on a Hilbert space and K is an algebraic operators commuting with T then $T + K$ satisfies Weyl's theorem. Note that Weyl's theorem is not generally transmitted to perturbation of operators satisfying Weyl's theorem.

6. a-Weyl's theorem

In this section we shall consider an approximate point version of Weyl's theorem introduced by Rakočević [**93**] .

Definition 4.49. *A bounded operator $T \in L(X)$ is said to satisfy* a-Weyl's theorem *if*

(49) $$\Delta^{\mathrm{a}}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T).$$

Theorem 4.50. *If a bounded operator $T \in L(X)$ satisfies a-Weyl's theorem then*

$$p^a_{00}(T) = \pi^a_{00}(T) = \Delta^{\mathrm{a}}(T).$$

Proof Suppose that T satisfies a-Weyl's theorem. By definition then $\Delta^{\mathrm{a}}(T) = \pi^a_{00}(T)$. We show now the equality $p^a_{00}(T) = \pi^a_{00}(T)$. It suffices to prove the inclusion $\pi^a_{00}(T) \subseteq p^a_{00}(T)$. Let λ be an arbitrary point of $\pi^a_{00}(T)$. Since λ is isolated in $\sigma_{\mathrm{a}}(T)$ then T has SVEP at λ and from the equality $\pi^a_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ we know that $\lambda I - T \in W_+(X)$. Hence $\lambda I - T \in \Phi_+(X)$ and the SVEP at λ implies that $p(\lambda I - T) < \infty$, so $\lambda \in p^a_{00}(T)$. ∎

Comparing Theorem 4.32 and Theorem 4.50 it immediately follows that for an operator a-Weyl's theorem implies a-Browder's theorem. It is not difficult to find an example of operator satisfying a-Browder's theorem but not a-Weyl's theorem. For instance, if $T \in L(\ell^2)$ is defined by

$$T(x_0, x_1, \dots) := (\frac{1}{2}x_1, \frac{1}{3}x_2, \dots) \quad \text{for all } (x_n) \in \ell^2,$$

then T is quasi-nilpotent, so has SVEP and consequently satisfies a-Browder's theorem. On the other hand T does not satisfy a-Weyl's theorem, since $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{uw}}(T) = \{0\}$ and $\pi_{00}^a(T) = \{0\}$. Note that the condition $\Delta^{\mathrm{a}}(T) = \emptyset$ does not ensure that a-Weyl's theorem holds.

To describe the operators which satisfy a-Weyl's theorem let us define

$$\Delta_{00}^{\mathrm{a}}(T) := \Delta^{\mathrm{a}}(T) \cup \pi_{00}^a(T).$$

Clearly, if $\Delta_{00}^{\mathrm{a}}(T) = \emptyset$ then the equalities (49) are satisfied, so this condition implies that a-Weyl's theorem holds. Theorem 4.44 has a companion for a-Weyl's theorem. In fact, a-Browder's theorem and a-Weyl's theorem are related by the following result:

Theorem 4.51. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T satisfies a-Weyl's theorem;*

(ii) *T satisfies a-Browder's theorem and $p_{00}^a(T) = \pi_{00}^a(T)$;*

(iii) *a-Browder's theorem holds for T and $(\lambda I - T)(X)$ is closed for all $\lambda \in \pi_{00}^a(T)$.*

(iv) *the map $\lambda \to \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}^{\mathrm{a}}(T)$;*

Proof We exclude the trivial case that $\Delta_{00}^{\mathrm{a}}(T) = \emptyset$.

(i) $\Leftrightarrow$ (ii) The implication (i) $\Rightarrow$ (ii) is clear, from Theorem 4.50 and Theorem 4.23. The implication (ii) $\Rightarrow$ (i) follows immediately from Theorem 4.32.

(i) $\Rightarrow$ (iii) If T satisfies a-Weyl's theorem then T obeys to a-Browder's theorem. Furthermore, $\pi_{00}^a(T) = p_{00}^a(T)$ by Theorem 4.50, so $\lambda I - T \in B_+(X)$ for all $\lambda \in \pi_{00}^a(T)$, and hence $(\lambda I - T)(X)$ is closed.

(iii) $\Rightarrow$ (ii) The condition $(\lambda I - T)(X)$ closed for all $\lambda \in \pi_{00}^a(T)$ entails that for these values of λ we have $\lambda I - T \in \Phi_+(X)$. Now, T has SVEP at every isolated point of $\sigma_{\mathrm{a}}(T)$, and in particular T has SVEP at every point of $\pi_{00}^a(T)$. By Theorem 2.51 and Theorem 2.45 it then follows that $p(\lambda I - T) < \infty$ for all $\lambda \in \pi_{00}^a(T)$, from which we deduce that $\pi_{00}^a(T) = p_{00}^a(T)$.

(i) $\Rightarrow$ (iv) By Theorem 4.50 we have that $\Delta_{00}^{\rm a}(T) = \Delta^{\rm a}(T)$ and T satisfies a-Browder's theorem, so by Theorem 4.35 the map $\lambda \to \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}^{\rm a}(T)$.

(iv) $\Rightarrow$ (iii) Suppose that $\lambda \to \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}^{\rm a}(T) = \Delta^{\rm a}(T) \cup \pi_{00}^a(T)$. The discontinuity at the points of $\Delta^{\rm a}(T)$ entails by Theorem 4.35 that T satisfies a-Browder's theorem. We show now that the discontinuity at a point λ_0 of $\pi_{00}^a(T)$ implies that $(\lambda_0 I - T)(X)$ is closed. In fact, if $\lambda_0 \in \pi_{00}^a(T)$ then $\lambda_0 \in \operatorname{iso} \sigma_{\rm a}(T)$ and $0 < \alpha(\lambda_0 I - T) < \infty$. Clearly, $\lambda I - T$ is injective in a punctured disc $\mathbb{D}$ centered at λ_0. Take $0 \neq x \in \ker(\lambda_0 I - T)$. If $\lambda \in \mathbb{D}$ then

$$\begin{aligned} \gamma(\lambda I - T) &\leq \frac{\|(\lambda I - T)x\|}{\text{dist } (x, \ker(\lambda I - T))} = \frac{\|(\lambda I - T)x\|}{\|x\|} \\ &= \frac{\|(\lambda I - T)x - (\lambda_0 I - T)x\|}{\|x\|} = |\lambda - \lambda_0|. \end{aligned}$$

¿From this estimate it follows that $\lim_{\lambda \to \lambda_0} \gamma(\lambda I - T) = 0 \neq \gamma(\lambda_0 I - T)$, so $(\lambda_0 I - T)(X)$ is closed. ■

¿From Theorem 4.35 and Theorem 4.51 we see that a-Browder's theorem and a-Weyl's theorem are equivalent to the discontinuity of the mapping $\lambda \to \gamma(\lambda I - T)$ at the points of two sets $\Delta^{\rm a}(T)$ and $\Delta_{00}^{\rm a}(T)$, respectively, with $\Delta_{00}^{\rm a}(T)$in general larger than $\Delta^{\rm a}(T)$. Comparing Theorem 4.51 and Theorem 4.35 one might expect that the discontinuity of the mapping $\lambda \to \ker(\lambda I - T)$ at every $\lambda \in \Delta_{00}^{\rm a}(T)$ is equivalent to a-Weyl's theorem for T. This does not work. In fact, by definition of $\pi_{00}^a(T)$ the map $\lambda \to \ker(\lambda I - T)$ is not continuous at every $\lambda \in \pi_{00}^a(T)$ for *all* operators $T \in L(X)$, since every $\lambda \in \pi_{00}^a(T)$ is an isolated point of $\sigma_{\rm a}(T)$.

Theorem 4.52. *If $T \in L(X)$ satisfies a-Weyl's theorem then T satisfies Weyl's theorem.*

Proof Suppose that a-Weyl's theorem holds for T. We claim that $\Delta_{00}^{\rm a}(T) \supseteq \Delta_{00}(T)$. We show first that $\Delta(T) \subseteq \Delta^{\rm a}(T)$. Let $\lambda \in \Delta(T) = \sigma(T) \setminus \sigma_{\rm w}(T)$. From the inclusion $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm w}(T)$ it follows that $\lambda \notin \sigma_{\rm uw}(T)$, and since $\lambda I - T$ is Weyl we also have $0 < \alpha(\lambda I - T) < \infty$, otherwise we would have $0 = \alpha(\lambda I - T) = \beta(\lambda I - T)$ and hence $\lambda \notin \sigma(T)$. Therefore $\lambda \in \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \Delta^{\rm a}(T)$, as desired.

We also have $\pi_{00}(T) \subseteq \pi^{\mathrm{a}}_{00}(T)$, thus, since T satisfies a-Weyl's theorem, we have

$$\Delta^{\mathrm{a}}_{00}(T) = \Delta^{\mathrm{a}}(T) \cup \pi^{a}_{00}(T) \supseteq \Delta(T) \cup \pi_{00}(T) = \Delta_{00}(T),$$

as claimed. By Theorem 4.51 the mapping $\lambda \to \gamma(\lambda I - T)$ is discontinuous at the points of $\Delta^{\mathrm{a}}_{00}(T)$ and hence at the points of $\Delta_{00}(T)$. By Theorem 4.44 we then conclude that Weyl's theorem holds for T. ∎

Example 4.53. We give now an example of operator $T \in L(X)$ which has SVEP, satisfies Weyl's theorem but does not satisfy a-Weyl's theorem. Let T be the hyponormal operator T given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbf{N})$. Then 0 is an isolated point of $\sigma_{\mathrm{a}}(T)$ and $0 \in \pi^{a}_{00}(T)$, while $0 \notin p^{a}_{00}(T)$, since $p(T) = p(R) = \infty$. Hence, T does not satisfy a-Weyl's theorem.

7. Property (w)

Another variant of Weyl's theorem has been introduced by Rakočević [**93**] and more recently studied in [**22**].

Definition 4.54. *A bounded operator $T \in L(X)$ is said to satisfy property (w) if*

$$\Delta^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T).$$

Unlike a-Weyl's theorem, the study of property (w) has been rather neglected, although, exactly like a-Weyl's theorem, property (w) implies Weyl's theorem (see next Theorem 4.57). A first result shows that property (w) entails a-Browder's theorem.

Theorem 4.55. *Suppose that $T \in L(X)$ satisfies property (w). Then a-Browder's holds for T.*

Proof By part (i) of Theorem 4.34 it suffices to show that T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$. Let $\lambda \notin \sigma_{\mathrm{uw}}(T)$. If $\lambda \notin \sigma_{\mathrm{a}}(T)$ then T has SVEP at λ, while if $\lambda \in \sigma_{\mathrm{a}}(T)$, then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ and hence $\lambda \in \operatorname{iso} \sigma(T)$, so also in this case T has SVEP at λ. ∎

Property (w) may be characterized in the following way:

Theorem 4.56. *If $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies property (w);*

(ii) *a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}(T)$.*

Proof (i) ⇒ (ii) By Theorem 4.55 we need only to prove the equality $p_{00}^a(T) = \pi_{00}(T)$. If $\lambda \in \pi_{00}(T) = \lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T \in W_+(X)$. Since λ is isolated in $\sigma(T)$ the SVEP of T at λ is equivalent to saying that $p(\lambda I - T) < \infty$, so $\lambda \in p_{00}^a(T)$. Hence $\pi_{00}(T) \subseteq p_{00}^a(T)$.

To show the opposite inclusion, suppose that $\lambda \in p_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. Since by Theorem 4.55 T satisfies a-Browder's theorem we have $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T)$, so $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. Therefore the equality $p_{00}^a(T) = \pi_{00}(T)$ is proved.

(ii) ⇒ (i) If $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ then a-Browder's theorem entails that $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T) = \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T)$ then $\lambda \in p_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. Hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. ■

Theorem 4.57. *If $T \in L(X)$ satisfies property (w) then Weyl's theorem holds for T.*

Proof Suppose that T satisfies property (w). By Theorem 4.55 T satisfies a-Browder's theorem and hence Browder's theorem. ¿From Theorem 4.44 we need only to prove that $\pi_{00}(T) = p_{00}(T)$. If $\lambda \in \pi_{00}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T)$, since $\alpha(\lambda I - T) > 0$, and from $\lambda \in \operatorname{iso}\sigma(T)$ we know that both T and T^* have SVEP at λ. From the equality $\pi_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ we see that $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and hence $\lambda I - T \in \Phi_+(X)$. The SVEP for T and T^* at λ, by Theorem 2.45 and Theorem 2.46, implies that $p(\lambda I - T) = q(\lambda I - T) < \infty$. From Theorem 1.21 we then obtain that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \in p_{00}(T)$. Hence $\pi_{00}(T) \subseteq p_{00}(T)$, and since the reverse inclusion holds for every $T \in L(X)$ we conclude that $\pi_{00}(T) = p_{00}(T)$. ■

The reverse of the result of Theorem 4.57 generally does not hold, see next Example 4.63. Define

$$\Lambda(T) := \{\lambda \in \Delta^{\mathrm{a}}(T) : \operatorname{ind}(\lambda I - T) < 0\}.$$

Clearly,

$$\Delta^{\mathrm{a}}(T) = \Delta(T) \cup \Lambda(T) \quad \text{and} \quad \Lambda(T) \cap \Delta(T) = \emptyset. \tag{50}$$

The next result relates Weyl's theorem and property (w).

Theorem 4.58. *If $T \in L(X)$ satisfies property (w) then $\Lambda(T) = \emptyset$. Moreover, the following statements are equivalent:*

(i) *T satisfies property (w);*

(ii) *T satisfies Weyl's theorem and $\Lambda(T) = \emptyset$;*

(iii) *T satisfies Weyl's theorem and $\Delta^{\rm a}(T) \subseteq \operatorname{iso}\sigma(T)$;*

(iv) *T satisfies Weyl's theorem and $\Delta^{\rm a}(T) \subseteq \partial\sigma(T)$, $\partial\sigma(T)$ the topological boundary of $\sigma(T)$;*

Proof Suppose that T satisfies property (w). Suppose that $\Lambda(T)$ is nonempty. Let $\lambda \in \Lambda(T)$. Then $\lambda \in \Delta^{\rm a}(T) = \pi_{00}(T)$, so λ is isolated in $\sigma(T)$ and hence T^* have SVEP at λ. Since $\lambda I - T \in \Phi_+(T)$, by Theorem 2.46, we have $q(\lambda I - T) < \infty$, and hence by Theorem 1.21 $\operatorname{ind}(\lambda I - T) \geq 0$, and this contradicts our assumption that $\operatorname{ind}(\lambda I - T) < 0$.

(i) $\Leftrightarrow$ (ii) The implication (i) $\Rightarrow$ (ii) is clear from the first part of the proof and from Theorem 4.57. Conversely, from the equality (50) we see that if $\Lambda(T) = \emptyset$ and T satisfies Weyl's theorem then we have $\Delta^{\rm a}(T) = \Delta(T) = \pi_{00}(T)$, so property (w) holds.

(iii) $\Rightarrow$ (ii) Suppose that T satisfies Weyl's theorem. If $\Delta^{\rm a}(T) \subseteq \operatorname{iso}\sigma(T)$, then both T and T^* have SVEP at every $\lambda \in \Delta^{\rm a}(T)$. As in the first part of the proof, this implies that $\operatorname{ind}(\lambda I - T) = 0$ for every $\lambda \in \Delta^{\rm a}(T)$, so $\Lambda(T) = \emptyset$. Hence property (w) holds for T.

(i) $\Rightarrow$ (iii) If property (w) holds then $\Delta^{\rm a}(T) = \pi_{00}(T) \subseteq \operatorname{iso}\sigma(T)$.

(iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (ii) Both T and T^* have SVEP at every point of $\partial\sigma(T) = \partial\sigma(T^*)$, so, by Theorem 2.45 and Theorem 2.46, $p(\lambda I - T) = q(\lambda I - T) < \infty$ for all $\lambda \in \Delta^{\rm a}(T)$. Finally, by Theorem 1.21 we conclude that $\operatorname{ind}(\lambda I - T) = 0$ for all $\lambda \in \Delta^{\rm a}(T)$, and hence $\Lambda(T) = \emptyset$. ∎

The condition $\Lambda(T) = \emptyset$ is satisfied by every Riesz operator $T \in L(X)$ on an infinite dimensional Banach space X, in particular by every compact operator. It is easily seen that Weyl's theorem holds for every compact operator having an infinite spectrum. However, Weyl's theorem may fail for a compact operator T, for an example see [**31**].

Corollary 4.59. *Suppose that $T \in L(X)$ is decomposable. Then T satisfies property (w) if and only if T satisfies Weyl's theorem.*

Proof If T is decomposable then both T and T^* have SVEP. This entails that $\lambda I - T$ has index 0 for every $\lambda \in \Delta^{\rm a}(T)$, and hence $\Lambda(T) = \emptyset$. The equivalence then follows from Theorem 4.58. ∎

As a consequence of Corollary 4.59 we have that for a bounded operator $T \in L(X)$ having totally disconnected spectrum then property

(w) and Weyl's theorem are equivalent.

Corollary 4.60. *If $T \in L(X)$ is generalized scalar then property (w) holds for both T and T^*.*

Proof Every generalized scalar operator T is decomposable and hence also the dual T^* is decomposable, see [**76**, Theorem 2.5.3]. Moreover, every generalized scalar operator has property $H(p)$, [**88**, Example 3], so Weyl's theorem holds for both T and T^*. By Corollary 4.59 it then follows that both T and T^* satisfy property (w). ∎

Example 4.61. Property (w), as well as Weyl's theorem, is not transmitted from T to its dual T^*. To see this, consider the weighted right shift $T \in L(\ell^2(\mathbb{N}))$, defined by

$$T(x_1, x_2, \dots) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$T^*(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}.$$

Both T and T^* are quasi-nilpotent, and hence are decomposable, T satisfies Weyl's theorem since $\pi_{00}(T) = p_{00}(T)$ and hence T has property (w), by Corollary 4.59. On the other hand, we have $\pi_{00}(T^*) = \{0\} \neq \sigma(T^*) \setminus \sigma_{\mathrm{w}}(T^*) = \emptyset$, so T^* does not satisfy Weyl's theorem. Since T^* is decomposable, by Corollary 4.59 then T^* does not satisfy property (w).

In the following diagram we resume the relationships between Weyl's theorems, a-Browder's theorem and property (w).

$$\begin{array}{ccc} \text{Property } (w) & \Rightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Leftarrow & a\text{-Weyl's theorem} \end{array}$$

The following examples show that property (w) and a-Weyl's theorem are in general not related. The first example provides an operator satisfying property (w) but not a-Weyl's theorem .

Example 4.62. Let T be the hyponormal operator T given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc in $\mathbb{C}$. Moreover, 0 is an isolated point of $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, Γ the unit circle of $\mathbb{C}$, and $0 \in \pi^a_{00}(T)$, while $0 \notin p^a_{00}(T) = \emptyset$, since $p(T) = p(R) = \infty$. Hence, by Theorem 4.51, T does not satisfy a-Weyl's theorem. On the other hand

$\pi_{00}(T) = \emptyset$, since $\sigma(T)$ has no isolated points, so $p_{00}^a(T) = \pi_{00}(T)$. Since every hyponormal operator has SVEP we also know that a-Browder's theorem holds for T, so from Theorem 4.56 we see that property (w) holds for T.

The following example provides an operator that satisfies a-Weyl theorem but not property (w)

Example 4.63. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N})$$

If $T := R \oplus U$ then $\sigma(T) = \mathbf{D}$ so $\operatorname{iso}\sigma(T) = \pi_{00}(T) = \emptyset$. Moreover, $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, $\sigma_{\mathrm{uw}}(T) = \Gamma$, so T does not satisfy property (w), since $\Delta_a(T) = \{0\}$. On the other hand we also have $\pi_{00}^a(T) = \{0\}$, so T satisfies a-Weyl's theorem.

We give now two sufficient conditions for which a-Weyl's theorem for T (respectively, T^*) implies property (w) for T (respectively, T^*).

Theorem 4.64. *If $T \in L(X)$ the following statements hold:*

(i) *If T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and T satisfies a-Weyl's theorem then property (w) holds for T.*

(ii) *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ and T^* satisfies a-Weyl's theorem then property (w) holds for T^*.*

Proof (i) We show first that if T^* has SVEP at every point $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then $\sigma_{\mathrm{a}}(T) \backslash \sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}(T)$. Let $\lambda \in \sigma_{\mathrm{a}}(T) \backslash \sigma_{\mathrm{uw}}(T)$. Since by part (iv) of Theorem 4.34 T satisfies a-Browder's theorem then $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, so $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T) \subseteq \pi_{00}^a(T)$, so $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$.

On the other hand, since $\lambda I - T \in B_+(X)$, by Theorem 2.52 we know that the SVEP for T^* at λ implies that $\lambda \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$. Therefore, $\lambda \in \operatorname{iso}\sigma(T)$. Since $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T$ has closed range we also have $0 < \alpha(\lambda I - T) < \infty$, and hence $\lambda \in \pi_{00}(T)$. This shows the inclusion $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}(T)$. To prove the opposite inclusion observe that a-Weyl's theorem for T entails that $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T) \supseteq \pi_{00}(T)$. Hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$, so property (w) holds for T.

(ii) Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$, and suppose that $\lambda \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*)$. By part (iii) of Theorem 4.34 then T^* satisfies a-Browder's theorem, so $\sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{ub}}(T^*)$ and by duality $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{lb}}(T)$. Hence $\lambda \in \sigma_{\mathrm{a}}(T^*) \backslash \sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{s}}(T) \backslash \sigma_{\mathrm{lb}}(T)$. Therefore $\lambda I - T \in B_-(X)$ and hence $q(\lambda I - T) < \infty$. This implies the SVEP for T^* at λ, or equivalently that $\lambda \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$. Since $\lambda I - T \in \Phi_-(X)$, our assumption of SVEP of T at λ entails also that $\lambda \in \operatorname{iso} \sigma_{\rm a}(T)$. Hence, $\lambda \in \operatorname{iso} \sigma(T) = \operatorname{iso} \sigma(T^*)$. Furthermore, since $\lambda \in \sigma_{\rm s}(T)$ and $\lambda I - T$ is semi-Fredholm we have $\alpha(\lambda I^* - T^*) = \beta(\lambda I - T) > 0$, so $\lambda \in \pi_{00}(T^*)$. This proves the inclusion $\sigma_{\rm a}(T^*) \setminus \sigma_{\rm uw}(T^*) \subseteq \pi_{00}(T^*)$. Finally, a-Weyl's theorem for T^* entails that

$$\sigma_{\rm a}(T^*) \setminus \sigma_{\rm uw}(T^*) = \pi_{00}^a(T^*) \supseteq \pi_{00}(T^*),$$

so that the equalities $\sigma_{\rm a}(T^*) \setminus \sigma_{\rm uw}(T^*) = \pi_{00}(T^*)$ hold, and hence property (w) holds for T^*. ■

The next result shows that Weyl's theorems and property (w) are equivalent in presence of SVEP.

Theorem 4.65. *If $T \in L(X)$ the following statements hold:*

(i) *If T^* has SVEP, the property (w) holds for T if and only if Weyl's theorem holds for T, and this is the case if and only if a-Weyl's theorem holds for T.*

(ii) *If T has SVEP, the property (w) holds for T^* if and only if Weyl's theorem holds for T^*, and this is the case if and only if a-Weyl's theorem holds for T^*.*

Proof (i) By Theorem 4.57 and part (i) of Theorem 4.64, for T we have the implications

$$a\text{-Weyl} \Rightarrow (\omega) \Rightarrow \text{Weyl} . \tag{51}$$

Assume now that T satisfies Weyl's theorem. The SVEP of T^* implies that $\sigma(T) = \sigma_{\rm a}(T)$, by Corollary 2.28, so

$$\pi_{00}^a(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_{\rm w}(T).$$

Furthermore, by Theorem 4.6 we also have $\sigma_{\rm w}(T) = \sigma_{\rm ub}(T)$ from which it follows that $\pi_{00}^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T) = p_{00}^a(T)$. Since the SVEP for T^* implies a-Browder's theorem for T we then conclude, by Theorem 4.51, that a-Weyl's theorem holds for T.

(ii) The argument is similar to that used in the proof of part (i). The implication (51) holds for T^* by Theorem 4.57 and part (ii) of Theorem 4.64. If T has SVEP then $\sigma(T^*) = \sigma(T) = \sigma_{\rm s}(T) = \sigma_{\rm a}(T^*)$, again by Corollary 2.28, and hence $\pi_{00}^a(T^*) = \pi_{00}(T^*)$. Moreover, by Theorem 4.6 we also have

$$\sigma_{\rm w}(T^*) = \sigma_{\rm w}(T) = \sigma_{\rm lb}(T) = \sigma_{\rm ub}(T^*),$$

from which it easily follows that $\pi^a_{00}(T^*) = p^a_{00}(T^*)$. The SVEP for T implies that T^* satisfies a-Browder's, so by part (ii) of Theorem 4.51 a-Weyl's theorem holds fo T^*. ∎

Remark 4.66. The operator T considered in Example 4.62 shows that in the statement (i) of Theorem 4.65 the SVEP for T^* cannot be replaced by the SVEP for T. Similarly, in the statement (ii) of Theorem 4.65 we cannot replace the SVEP for T with the SVEP for T^*. For instance, let $0 < \varepsilon < 1$ and define $T \in L(\ell^2(\mathbb{N}))$ by

$$T(x_1, x_2, \dots) := (\varepsilon x_1, 0, x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then $\sigma_{\mathrm{a}}(T^*) = \Gamma \cup \{\varepsilon\}$, and hence $\operatorname{int} \sigma_{\mathrm{a}}(T^*) = \emptyset$, which implies that T^* has SVEP. Moreover, $\sigma_{\mathrm{uw}}(T^*) = \Gamma$, $\pi^a_{00}(T^*) = \{\varepsilon\}$, so a-Weyl's theorem holds for T^*. On the other hand, it is easy to see that $\pi_{00}(T^*) = \emptyset$, so property (w) does not hold for T^*.

In the case of operators defined on Hilbert spaces instead of the dual T^* it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$. However, some of the basic results established in the previous sections for T^* are also true for the adjoint T'. In fact, by means of the classical Fréchet- Riesz representation theorem we know that if U is the conjugate-linear isometry that associates to each $y \in H$ the linear form $x \to \langle x, y\rangle$ then $UT' = T^*U$. From this equality, by using an argument similar to that used in the proof of Theorem 2.17, we easily obtain that

$$T' \text{ has SVEP at } \lambda_0 \Leftrightarrow T^* \text{ has SVEP at } \lambda_0,$$

In the case that the Hilbert adjoint T' of T has property $H(p)$ we have the following interesting result.

Corollary 4.67. *If T' has property $H(p)$ then property (w) (or equivalently, a-Weyl's theorem) holds for $f(T)$ for every f analytic on an open set containing $\sigma(T)$. In particular, if T' is generalized scalar then property (w) holds for $f(T)$.*

Proof If T' has property (H_p) then Weyls theorem holds for T, see [**4**, Theorem 4.1]. Moreover, T' has SVEP, and, as observed before, this entails that also T^* has SVEP. Therefore, by Theorem 4.18 $f(T)^* = f(T^*)$ has SVEP so part (i) of Theorem 4.65 applies. Property (w) and a-Weyl's theorem are equivalent, again by Theorem 4.65. ∎

¿From Corollary 4.67 it then follows that if T' belongs to each one of the classes of operators mentioned in Theorem4.48 then property (w), or equivalently a-Weyl's theorem, holds for $f(T)$.

A similar result holds for algebraically paranormal operators. If T' is algebraically paranormal then, see Theorem 2.40 of [**4**], a-Weyl's theorem holds for $f(T)$. Moreover, T' has SVEP and hence $f(T)^* = f(T^*)$ has SVEP by Theorem 4.18, so by Theorem 4.65 $f(T)$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.

An operator $T \in L(X)$ is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$, or equivalently $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. An operator $T \in L(X)$ is said to be *a-polaroid* if every isolated point of $\sigma_{\rm a}(T)$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$, or equivalently $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Clearly,

$$T \ a\text{-polaroid} \Rightarrow T \text{ polaroid}.$$

and the opposite implication is not generally true.

For a-polaroid operators the equivalence between a-Weyl's theorem and property (w) is true without assuming the SVEP for T^*.

Theorem 4.68. *Suppose that T is a-polaroid. Then a-Weyl's theorem holds for T if and only if T satisfies property (w).*

Proof Note first that if T is a-polaroid then $\pi^a_{00}(T) = p_{00}(T)$. In fact, if $\lambda \in \pi^a_{00}(T)$ then λ is isolated in $\sigma_{\rm a}(T)$ and hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. Moreover, $\alpha(\lambda I - T) < \infty$, so by Theorem 1.21 it follows that $\beta(\lambda I - T)$ is also finite, thus $\lambda \in p_{00}(T)$. This shows that $\pi^a_{00}(T) \subseteq p_{00}(T)$, and consequently by Lemma 4.40 we have $\pi^a_{00}(T) = p_{00}(T)$.

Now, if T satisfies a-Weyl's theorem then $\Delta^{\rm a}(T) = \pi^a_{00}(T) = p_{00}(T)$, and since Weyl's theorem holds for T we also have by Theorem 4.44 that $p_{00}(T) = \pi_{00}(T)$. Hence property (w) holds for T.

Conversely, if T satisfies property (w) then $\Delta^{\rm a}(T) = \pi_{00}(T)$. Since by Theorem 4.57 T satisfies Weyl's theorem we also have, by Theorem 4.44, $\pi_{00}(T) = p_{00}(T) = \pi^a_{00}(T)$, so T satisfies a-Weyl's theorem. ∎

The last theorem implies that property (w) holds for every multiplier $T \in M(A)$ of a commutative semi-simple regular Tauberian Banach algebra A, and in particular for every convolution operator on $L^1(G)$, where G is a compact Abelian group (see Chapter 4 of [**1**] for definitions and details). In fact a-Weyl's theorem holds for T [**24**], and by Theorem 5.54 and Theorem 4.36 of [**1**], every multiplier$T$ is a-polaroid.

The operator defined in Example 4.63 shows that a similar result to that of Theorem 4.68 does not hold for polaroid operators, i.e. if

$T \in L(X)$ is polaroid Weyl's theorem for T and property (w) for T in general are not equivalent. However, we have

Theorem 4.69. *Suppose that $T \in L(X)$. Then the following statements hold*

(i)*If T is polaroid and T has SVEP then property (w) holds for T^*.*

(i)*If T is polaroid and T^* has SVEP then property (w) holds for T.*

Proof (i) By Theorem 4.65 it suffices to show that Weyl's theorem holds for T^*. The SVEP ensures that Browder's theorem holds for T^*. We prove that $\pi_{00}(T^*) = p_{00}(T^*)$. Let $\lambda \in \pi_{00}(T^*)$. Then $\lambda \in \operatorname{iso}\sigma(T^*) = \operatorname{iso}\sigma(T)$ and the polaroid assumption implies that λ is a pole of the resolvent, or equivalently $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. If P denotes the spectral projection associated with $\{\lambda\}$, by Theorem 2.9 we have $(\lambda I - T)^p(X) = \ker P$ so $(\lambda I - T)^p(X)$ is closed, and hence also $(\lambda I^* - T^*)^p(X^*)$ is closed. Since $\lambda \in \pi_{00}(T^*)$ then $\alpha(\lambda I^* - T^*) < \infty$ and this implies $\alpha(\lambda I^* - T^*)^p < \infty$, from which we conclude that $(\lambda I^* - T^*)^p \in \Phi_+(X^*)$, hence $\lambda^* I - T^* \in \Phi_+(X^*)$, and consequently $\lambda I - T \in \Phi_-(X)$. Therefore $\beta(\lambda I - T) < \infty$ and since $p(\lambda I - T) = q(\lambda I - T) < \infty$ by Theorem 1.21 we then conclude that $\alpha(\lambda I - T) < \infty$. Hence $\lambda \in p_{00}(T) = p_{00}(T^*)$. This proves that $\pi_{00}(T^*) \subseteq p_{00}(T^*)$, and since the opposite inclusion is satisfied by every operator we may conclude that $\pi_{00}(T^*) = p_{00}(T^*)$. By Theorem 4.44 then T^* satisfies Weyl's theorem.

(ii) The SVEP for T^* implies that Browder's theorem holds for T. Again by Theorem 4.65 it suffices to show that T satisfies Weyl's theorem, and hence by Theorem 4.44 we need only to prove that $\pi_{00}(T) = p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and since T is polaroid then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) < \infty$ we then have $\beta(\lambda I - T) < \infty$ and hence $\lambda \in p_{00}(T)$. Hence $\pi_{00}(T) \subseteq p_{00}(T)$ and hence we can conclude that $\pi_{00}(T) = p_{00}(T)$. ■

Part (i) of Theorem 4.69 shows that the dual T^* of a multiplier $T \in M(A)$ of a commutative semi-simple Banach algebra A has property (w), since every multiplier $T \in M(A)$ of a commutative semi-simple Banach algebra satisfies Weyl's theorem and is polaroid, see Theorem 4.36 of [**1**].

8. Further and recent developments

It is natural to ask if Weyl's theorems or property (w) is transmitted from a bounded operator $T \in L(X)$ to some perturbation $T + K$. The following result has been first proved by Oberai [**87**].

Theorem 4.70. *Weyl's theorem is transmitted from $T \in L(X)$ to $T + N$ when N is a nilpotent operator commuting with T.*

Proof The assertion follows from Theorem 4.46 and from the fact the $\sigma(T) = \sigma(T + N)$, see Lemma 2.75, and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T + N)$, see Theorem 4.7. ∎

The following example shows that Oberai's result does not hold if we do not assume that the nilpotent operator N commutes with T, so we see that also for a simple case the condition of commutativity is required.

Example 4.71. Let $X := \ell^2(\mathbb{N})$ and T and N be defined by

$$T(x_1, x_2, \dots,) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

and

$$N(x_1, x_2, \dots,) := (0, -\frac{x_1}{2}, 0, 0, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

Clearly, N is a nilpotent operator, and T is a quasi-nilpotent operator satisfying Weyl's theorem. On the other hand, it is easily seen that $0 \in \pi_{00}(T + N)$ and $0 \notin \sigma(T + N) \setminus \sigma_{\mathrm{w}}(T + N)$, so that $T + N$ does not satisfies Weyl's theorem.

Note that the operator N in Example 4.71 is also a finite rank operator not commuting with T. In general, Weyl's theorem is also not transmitted under commuting finite rank perturbation, as the following example shows.

Example 4.72. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasi-nilpotent operator, and let $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined :

$$U(x_1, x_2, \dots,) := (-x_1, 0, 0, \dots), \quad \text{with } (x_n) \in \ell^2(\mathbb{N}).$$

Define on $X := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ the operators T and K by

$$T := I \oplus S \quad \text{and} \quad K := U \oplus 0$$

Clearly, K is a finite rank operator and $KT = TK$. It is easy to check that

$$\sigma(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{a}}(T) = \{0, 1\}.$$

Now, both T and T^* have SVEP, since $\sigma(T) = \sigma(T^*)$ is finite. Moreover, $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\rm w}(T) = \emptyset$, so T satisfies Weyl's theorem.
On the other hand,

$$\sigma(T + K) = \sigma_{\rm w}(T + K) = \{0, 1\},$$

and $\pi_{00}(T+K) = \{0\}$, so that Weyl's theorem does not hold for $T+K$.

Definition 4.73. *A bounded operator $T \in L(X)$ is said to be* isoloid *if every isolated point λ of the spectrum is an eigenvalue of T. $T \in L(X)$ is said to be* finite-isoloid *if every isolated spectral point λ is an eigenvalue having finite multiplicity, i.e. $0 < \alpha(\lambda I - T) < \infty$.*

A result of W. Y. Lee and S. H. Lee [**77**] shows that Weyl's theorem for an isoloid operator is preserved by perturbations of commuting finite rank operators. This result has been generalized by Oudghiri [**89**] as follows:

Theorem 4.74. *If $T \in L(X)$ is an isoloid operator which satisfies Weyl's theorem, if $ST = TS$, $S \in L(X)$, and there exists $n \in \mathbb{N}$ such that S^n is finite-dimensional, then $T + S$ satisfies Weyl's theorem.*

More recently, Y. M. Han and W. Y. Lee in [**65**] have shown that in the case of Hilbert spaces if T is a finite-isoloid operator which satisfies Weyl's theorem and if S a compact operator commuting with T then also $T + S$ satisfies Weyl's theorem. Again, Oudghiri [**89**] has shown that we have much more:

Theorem 4.75. *If $T \in L(X)$, X a Banach space, is a finite-isoloid operator which satisfies Weyl's theorem and if K is a Riesz operator commuting with T then also $T + K$ satisfies Weyl's theorem.*

Recall that a bounded operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. Note that if T^n is finite-dimensional for some $n \in \mathbb{N}$ then T is algebraic. The following two results show that Weyl's theorem survives under commuting algebraic perturbations in some special cases.

Theorem 4.76. [**89**] *Suppose that $T \in L(X)$ has property $H(p)$, K algebraic and $TK = KT$. Then Weyl's theorem holds for $T + K$*

As already observed the property $H(p)$ may fail for paranormal operators, in particular fails for quasi-hyponormal operators [**21**]. However, to these operators we can apply the next result to this case.

Theorem 4.77. [**21**] *Suppose that $T \in L(H)$ is paranormal, K algebraic and $TK = KT$. Then Weyl's theorem holds for $T + K$.*

As observed above every finite-dimensional operator is algebraic. If T satisfies $H(p)$, or is paranormal, then T is isoloid (see [**4**]), so that the result of Theorem 4.74 in these special cases also follows from Theorem 4.76 and Theorem 4.77.

In the perturbation theory the "commutative" condition is rather rigid. On the other hand, it is known that without the commutativity, the spectrum can however undergo a large change under even rank one perturbations. However, we have the following result due to Y. M. Han and W. Y. Lee [**65**].

Theorem 4.78. *Suppose that $T \in L(H)$ is a finite-isoloid operator which satisfies Weyl's theorem. If $\sigma(T)$ has no holes (bounded components of the complement) and has at most finitely many isolated points then Weyl's theorem holds for $T + K$, where $K \in L(H)$ is either a compact or quasi-nilpotent operator commuting with T modulo the compact operators.*

The result below follows immediately from Theorem 4.78

Corollary 4.79. *Suppose that $T \in L(H)$ satisfies Weyl's theorem. If $\sigma(T)$ has no holes and has at most finitely many isolated points then Weyl's theorem holds for $T + K$ for every compact operator K.*

Corollary 4.79 applies to Toeplitz operators and not quasi-nilpotent unilateral weighted shifts, see [**65**].

It is easy to find an example of an operator such that a-Weyl's theorem holds for T while there is a commuting finite rank operator K such that a-Weyl's theorem fails for $T + K$.

Example 4.80. Let Q be any injective quasi-nilpotent operator on a Banach space X. Define $T := Q \oplus I$ on $X \oplus X$. Clearly, T satisfies a-Weyl's theorem. Take $P \in L(X)$ any finite rank projection and set $K := 0 \oplus (-P)$. Then $TK = KT$ and $0 \in \pi^a_{00}(T + K) \cap \sigma_{\rm uw}(T + K)$. Clearly, $0 \notin \sigma_{\rm a}(T+K) \setminus \sigma_{\rm uw}(T+K)$, so that $\sigma_{\rm a}(T+K) \setminus \sigma_{\rm uw}(T+K) \neq \pi^a_{00}(T)$ and hence a-Weyl's theorem does not holds for $T + K$.

To see when a-Weyls theorem is transmitted under some perturbations we need to introduce some definitions. A bounded operator $T \in L(X)$ is said to be *a-isoloid* if every isolated point of the approximate point spectrum $\sigma_{\rm a}(T)$ is an eigenvalue. $T \in L(X)$ is said to be *finite a-isoloid* if every isolated point $\sigma_{\rm a}(T)$ is an eigenvalue having finite multiplicity. The following two results are due to Oudghiri [**90**].

Theorem 4.81. [**90**] *If $T \in L(X)$ is an a-isoloid operator which satisfies a-Weyl's theorem, if $ST = TS$, $S \in L(X)$, and there exists $n \in \mathbb{N}$ such that S^n is finite-dimensional, then $T + S$ satisfies a-Weyl's theorem.*

Theorem 4.81 extends a result of D. S. Djordjević [**47**] , where a-Weyl's theorem was proved for $T + S$ when S is a finite rank operator commuting with T. In the case of finite a-isoloid operators we can say much more:

Theorem 4.82. [**90**] *If $T \in L(X)$ is a finite a-isoloid operator which satisfies a-Weyl's theorem and if K a Riesz operator commuting with T then also $T + K$ satisfies a-Weyl's theorem.*

In particular, Theorem 4.82 applies to compact perturbations $T+K$.

We have seen that property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem. Property (w) is preserved by commuting nilpotent perturbations in the case that T is a-isoloid.

Theorem 4.83. [**7**] *Suppose that $T \in L(X)$ is a-isoloid. If T satisfies property (w) and N is nilpotent operator that commutes with T then $T + N$ satisfies property (w).*

Property (w) is not preserved by commuting finite rank perturbations K, also in the case that T is a-isoloid. However, we have:

Theorem 4.84. [**7**] *Suppose that $T \in L(X)$ is a-isoloid and K is a finite rank operator that commutes with T. If T satisfies property (w) and $\sigma_{\rm a}(T + K) = \sigma_{\rm a}(T)$ then $T + K$ satisfies property (w).*

Example 4.85. In general, property (w) is not transmitted from T to a quasi-nilpotent perturbation $T + Q$. For instance, take $T = 0$, and $Q \in L(\ell^2(\mathbb{N}))$ defined by

$$Q(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N},$$

Then Q is quasi-nilpotent and $\{0\} = \pi_{00}(Q) \neq \sigma_{\rm a}(Q) \setminus \sigma_{\rm uw}(Q) = \emptyset$. Hence T satisfies property (w) but $T + Q = Q$ fails this property.

Note that in the Example 4.85 Q is not injective. We want to show that property (w), as well as a-Weyl's theorem, holds for T if T commutes with an injective quasi-nilpotent operator. We need first a preliminary result.

Lemma 4.86. *Let $T \in L(X)$ be such that $\alpha(T) < \infty$. Suppose that there exists an injective quasi-nilpotent operator $Q \in L(X)$ such that $TQ = QT$. Then T is injective.*

Proof Set $Y := \ker T$. Clearly, Y is invariant under Q and the restriction $(\lambda I - Q)|Y$ is injective for all $\lambda \neq 0$. Since Y is finite-dimensional then $(\lambda I - Q)|Y$ is also surjective, thus $\sigma(Q|Y) \subseteq \{0\}$. On the other hand, from assumption we know that $Q|Y$ is injective and hence $Q|Y$ is surjective, so $\sigma(Q|Y) = \emptyset$, from which we conclude that $Y = \{0\}$. ∎

Theorem 4.87. *Suppose that for $T \in L(X)$ there exists an injective quasi-nilpotent Q operator commuting with T. Then both T and $T + Q$ satisfy property (w) and a-Weyl's theorem (and, consequently, Weyl's theorem).*

Proof We show first property (w) for T. It is evident, by Lemma 4.86, that $\pi_{00}(T)$ is empty.

Suppose that $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ is not empty and let $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. Since $\lambda I - T \in W_+(X)$ then $\alpha(\lambda I - T) < \infty$ and $\lambda I - T$ has closed range. Since $\lambda I - T$ commutes with Q it then follows, by Lemma 4.86, that $\lambda I - T$ is injective, so $\lambda \notin \sigma_{\mathrm{a}}(T)$, a contradiction. Therefore, also $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ is empty.

Property (w) for $T + Q$ is clear, since also $T + Q$ commutes with Q.

To show that a-Weyl's theorem holds for T observe that by Lemma 4.86, also $\pi^a_{00}(T)$ is empty, hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T) = \emptyset$. Analogously, a-Weyl's theorem also holds for $T + Q$, since $T + Q$ commutes with Q. ∎

Property (w) is preserved under finite rank perturbations or nilpotent perturbations commuting with T in the case that $T^* \in L(X)$ has property $H(p)$ or T^* is algebraically paranormal. This is a consequence of the next two theorems, taking into account that finite rank operators and nilpotent are algebraic.

Remark 4.88. In the case of an operator defined on a Hilbert space H instead of the dual T^* of $T \in L(H)$ it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$. However, we have [**4**]

$$T' \text{ has SVEP} \Leftrightarrow T^* \text{ has SVEP},$$

so that the result of Corollary 4.105 holds if we suppose that T' has SVEP.

Theorem 4.89. [**11**] *Suppose that $T \in L(H)$, H a Hilbert space, and let K be an algebraic operator commuting with T. The following statements hold:*

(i) *if T is algebraically paranormal then property (w) holds for $T' + K'$.*

(ii) *if T' is algebraically paranormal then property (w) holds for $T + K$.*

An analogous result holds for $H(p)$ operators defined on Banach spaces:

Theorem 4.90. [**11**] *Suppose that $T \in L(X)$ and K is an algebraic operator commuting with T.*

(i) *if $T \in H(p)$ then property (w) holds for $T^* + K^*$.*

(ii) *if $T^* \in H(p)$ then property (w) holds for $T + K$.*

Browder's and Weyl's theorems admit a generalization in the sense of semi B-Fredholm operators. Recall that a bounded operator $T \in L(X)$ is said to be *B-Browder* (resp. *upper semi B-Browder*, *lower semi B-Browder*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder). The respective B-Browder spectra are denoted by $\sigma_{\rm bb}(T)$, $\sigma_{\rm usbb}(T)$ and $\sigma_{\rm lsbb}(T)$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* (resp. *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (resp. upper semi-Weyl, lower semi-Weyl). The respective B-Weyl spectra are denoted by $\sigma_{\rm bw}(T)$, $\sigma_{\rm usbw}(T)$ and $\sigma_{\rm lsbw}(T)$. Finally, define the *left Drazin spectrum* as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the *right Drazin spectrum* as

$$\sigma_{\rm rd}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

and the *Drazin spectrum* is defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$.

By Theorem 2.92 we have

Theorem 4.91. *For every $T \in L(X)$ we have*

$$\sigma_{\rm usbb}(T) = \sigma_{\rm ld}(T), \quad \sigma_{\rm lsbb}(T) = \sigma_{\rm rd}(T), \quad \sigma_{\rm bb}(T) = \sigma_{\rm d}(T).$$

The results of following two theorems are analogous to those established in Theorem 4.5 and Theorem 4.6.

Theorem 4.92. [**8**] *For a bounded operator $T \in L(X)$ the following equalities hold:*

(i) $\sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{usbw}}(T) \cup acc\,\sigma_{\mathrm{a}}(T)$.

(ii) $\sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{lsbw}}(T) \cup acc\,\sigma_{\mathrm{s}}(T)$.

(iii) $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bw}}(T) \cup acc\,\sigma(T)$.

If T or T^* has SVEP some of these spectra coincide.

Theorem 4.93. [**8**] *Suppose that $T \in L(X)$. Then the following statements hold:*

(i) *If T has SVEP then*

$$\sigma_{\mathrm{usbw}}(T) = \sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T). \tag{52}$$

(ii) *If T^* ha SVEP then*

$$\sigma_{\mathrm{lsbw}}(T) = \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{53}$$

(iii) *If both T and T^* ha SVEP then*

$$\sigma_{\mathrm{usbw}}(T) = \sigma_{\mathrm{lsbw}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{54}$$

A consequence of Theorem 4.93 is that the spectral mapping theorem holds for $\sigma_{\mathrm{usbw}}(T)$ in the case that T^* has SVEP.

Theorem 4.94. [**13**] *If $T \in L(X)$ and T^* has SVEP then the spectral mapping theorem holds for $\sigma_{\mathrm{usbw}}(T)$, i.e.*

$$f(\sigma_{\mathrm{usbw}}(T)) = \sigma_{\mathrm{usbw}}(f(T))$$

for each $f \in \mathcal{H}(\sigma(T))$.

In the case that T has SVEP we require that f is not constant on each of the components of its domain.

Theorem 4.95. [**13**] *Suppose that T has SVEP and that $f \in \mathcal{H}(\sigma(T))$ is not constant on each of the components of its domain. Then $f(\sigma_{\mathrm{usbw}}(T)) = \sigma_{\mathrm{usbw}}(f(T))$.*

Define $\Pi_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$ and $\Pi^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Moreover, set

$$\Delta_{\mathrm{b}}(T) := \sigma(T) \setminus \sigma_{\mathrm{bw}}(T) \quad \text{and} \quad \Delta_{\mathrm{b}}^{\mathrm{a}}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbw}}(T).$$

Definition 4.96. [**35**] *$T \in L(X)$ is said to satisfy* generalized Browder's theorem *if $\Delta_{\mathrm{b}}(T) = \Pi_{00}(T)$, or equivalently $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$. A bounded operator $T \in L(X)$ is said to satisfy generalized a-Browder's theorem if $\Delta_{\mathrm{b}}^{\mathrm{a}}(T) = \Pi^{a}(T)$, or equivalently $\sigma_{\mathrm{usbw}}(T) = \sigma_{\mathrm{ld}}(T)$.*

Clearly, by Theorem 2.92

$$\text{generalized Browder's theorem holds for } T \Leftrightarrow \sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bw}}(T), \tag{55}$$

or equivalently, by Theorem 4.92,

$$\text{generalized Browder's theorem holds for } T \Leftrightarrow \operatorname{acc} \sigma(T) \subseteq \sigma_{\mathrm{bw}}(T). \tag{56}$$

Analogously,

$$\text{generalized } a\text{- Browder's theorem holds for } T \Leftrightarrow \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{usbw}}(T), \tag{57}$$

or equivalently, by Theorem 4.92,

$$\text{generalized } a\text{-Browder's theorem holds for } T \Leftrightarrow \operatorname{acc} \sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{usbw}}(T). \tag{58}$$

Theorem 4.97. *([**17**], [**13**]) Let $T \in L(X)$. Then the following statements hold:*

(i) *T satisfies generalized Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{bw}}(T)$*

(ii) *T satisfies generalized a-Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbw}}(T)$*

Since $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$ and $\sigma_{\mathrm{usbw}}(T) \subseteq \sigma_{\mathrm{uw}}(T)$ for all $T \in L(X)$, by Theorem 4.97 and Theorem 4.23 we readily obtain

$$\text{generalized Browder's theorem for } T \Rightarrow \text{Browder's theorem for } T.$$

and

$$\text{generalized } a\text{-Browder's theorem for } T \Rightarrow a\text{-Browder's theorem for } T.$$

The main result of a very recent paper [**27**] proves that Browder's theorems and generalized Browder's theorem (respectively, a-Browder's theorems and generalized a-Browder's theorem) are equivalent, see also [**8**] for a shorter proof.

Theorem 4.98. *For every $T \in L(X)$ the following equivalences hold:*

(i) *Browder's theorem for T and generalized Browder's theorem for T are equivalent.*

(ii) *a-Browder's theorem for T and generalized a-Browder's theorem for T are equivalent.*

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$, let $f(T)$ be defined by means of the classical functional calculus.

Theorem 4.99. *If either T or T^* has SVEP and $f \in \mathcal{H}(\sigma(T))$ then the spectral mapping theorem holds for $\sigma_{\rm bw}(T)$, i.e.*

$$\sigma_{\rm bw}(f(T)) = f(\sigma_{\rm bw}(T)) \quad \textit{for all } f \in \mathcal{H}(\sigma(T)).$$

Define

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}.$$

It is easily seen that

$$\Pi_{00}(T) \subseteq E(T) \quad \text{for all } T \in L(X).$$

Also Weyl's theorem may be generalized in the semi B-Fredholm sense of Berkani :

Definition 4.100. *A bounded operator $T \in L(X)$ is said to satisfy* generalized Weyl's theorem *if $\Delta(T) = \sigma(T) \setminus \sigma_{\rm uw}(T) = E(T)$.*

Generalized Weyl's theorem has been recently studied by several authors, see [**32**], [**37**], [**35**], [**107**], and its relevance is due to the fact that it implies the classical Weyl's theorem . Define

$$\Delta_1(T) := \Delta_{\rm b}(T) \cup E(T).$$

Theorem 4.101. [**17**] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies generalized Weyl's theorem;*

(ii) *T satisfied generalized Browder's theorem and $E(T) = \Pi_{00}(T)$;*

(iii) *For every $\lambda \in \Delta_1(T)$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$.*

Define

$$E^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\rm a}(T) : 0 < \alpha(\lambda I - T)\}.$$

Definition 4.102. *A bounded operator $T \in L(X)$ is said to satisfies* generalized a-Weyl's theorem *if $\Delta^a(T) = E^a(T)$.*

Define

$$\Delta_1^a(T) := \Delta_{\rm b}^{\rm a}(T) \cup E^a(T).$$

The following characterization of generalized a-Weyl's theorem has been proved in a recent paper of Aiena and Miller [**13**] .

Theorem 4.103. [**13**] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies generalized a-Weyl's theorem;*

(ii) *T satisfies generalized a-Browder's theorem and the equality $E^a(T) = \Pi^a(T)$ holds;*

(iii) *For every $\lambda \in \Delta_1^a(T)$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq p$.*

Since

$$\Delta_1(T) = \Delta_{\rm b}(T) \cup E(T) \subseteq \Delta_{\rm b}^{\rm a}(T) \cup E^a(T) = \Delta_1^a(T)$$

we easily deduce from Theorem 4.103 and Theorem 4.101 the following implication:

generalized a-Weyls's theorem for $T \Rightarrow$ generalized Weyl's theorem for T.

In a very important situation the previous implication may be reversed:

Theorem 4.104. [**13**] *If T^* has SVEP then generalized a-Weyl's theorem holds for T if and only if generalized Weyl's theorem holds for T.*

Corollary 4.105. *If T^* has SVEP and T is polaroid then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof By Theorem 1.20 of [**17**] $f(T)$ satisfies generalized Weyl's theorem. By [**1**, Theorem 2.40] $f(T^*) = f(T)^*$ has SVEP. ∎

The following result has been proved in [**13**].

Theorem 4.106. *If $T \in L(X)$, X a Banach space, and T^* satisfies the condition $H(p)$, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T)$. Analogously, if $T' \in L(H)$, H a Hilbert space, is algebraically paranormal then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

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